A theorem is proved which establishes a connection among Laplace, Kantorovich-Lebedev, Meier, and the generalized Mehler-Fock transforms. Some improper integrals are calculated by using this theorem and its immediate generalization.

A theorem was proved in [1] which established a connection among the Laplace, Kantorovich-Lebedev, Mehler-Fock, and Meier K-transforms. The purpose of the present note is an immediate generalization of this theorem to the case when the Mehler-Fock transform is replaced by the generalized Mehler-Fock transform in the sense of [2]. The following theorem holds:

**THEOREM 1.** Let the function \( f(x) \) satisfy the following conditions:

1) \[ \int_0^\infty |f(x)| \, dx < \infty; \]

2) the integral

\[ \int_0^\infty I(y) P_{n; \nu + i \tau}^{m; \mu}(y)(y^2 - 1)^{-\nu} \, dy, \]

where \( I(y) = \int_0^\infty \exp(-xy)f(x) \, dx, \) \( 0 < \tau < \infty, \) \( \Re m < 1/2, \) \( P_{n; \nu + i \tau}^{m; \mu}(x) \) is the associated Legendre function, converges absolutely. Then the generalized Mehler-Fock transform of the function

\[ F(R) = \sqrt{\frac{\pi}{2}} \int_0^\infty \exp(-Rx) \, f(x) \, dx \]

coincides with the Kantorovich-Lebedev transform of the function \( \varphi_M(x) = f(x)x^{m-1/2} \) and is a special case of the Meier K-transform [3, 4],

\[ \int_0^\infty \varphi_M(x) K_v(xy)(xy)^{\frac{\nu}{2}} \, dx \]

of the function \( \varphi_M(x) = x^{m-1/2}f(x) \) for \( y = 1, \nu = i\tau. \)

**Proof.** The absolute convergence of the integral (2) for \( R = 0 \) follows from the condition 1).

Let us establish the absolute convergence of the integral

\[ \int_0^\infty e^{-Rx} P_{n; \nu + i \tau}^{m; \mu}(R)(R^2 - 1)^{-\nu} \, dR. \]

From the integral representation 8.715.1 in [5] for the function \( P_{\nu}^{m}(x) \) for \( \Re m < 1/2, x > 1, \) we have the following estimate:

\[ |P_{n; \nu + i \tau}^{m; \mu}(x)| \leq \frac{\Gamma(\frac{\nu}{2} - \Re m)}{\Gamma(\frac{\nu}{2} - m)} P_{n; \nu + i \tau}^{m; \mu}(x). \]

Comparing (4) and (5), we obtain
This proves the absolute convergence of the integral (4) for \( \Re m < \frac{1}{2}, x > 1 \). Furthermore, the absolute convergence of the integral

\[
\int_{-\infty}^{\infty} F(R) P_{n+1}^{m}(R) (R^2 - 1)^{-m/2} dR
\]

follows from the condition 2). Thus, substituting the expression (2) for \( F(R) \) into (7), we can change the order of integration. Using the formula 7.141.5 in [5], we obtain the relation

\[
\int_{0}^{\infty} F(R) P_{n+1}^{m}(R) (R^2 - 1)^{-m/2} dR = \int_{0}^{\infty} f(x) x^{m-\frac{1}{2}} K_{\nu}(x) dx,
\]

where \( \Re m < \frac{1}{2} \). The last integral in (8) is itself the Kantorovich-Lebedev transform of the function \( \phi_{m}(x) = f(x)x^{m-\frac{1}{2}} \).

To prove the rest of the theorem it is sufficient to represent the integral

\[
\int_{0}^{\infty} f(x) x^{m-\frac{1}{2}} K_{\nu}(x) dx
\]

in the form

\[
\int_{0}^{\infty} \psi_{m}(x) K_{\nu}(x) x^{\frac{\nu}{2}} dx,
\]

where \( \psi_{m}(x) = x^{m-\frac{1}{2}} f(x) \). The integral (9) is itself a special case of the K-transform for \( \nu = 1, \nu = i \tau \) (0 \( \leq \tau < \infty \)) of the function \( \phi_{m}(x) \). This proves the theorem.

The theorem of the note [1] is the special case \( m = 0 \) of the above theorem. Theorem 1 in turn can be generalized to an arbitrary complex index \( \nu \) of the function \( P_{\nu}^{m}(x) \). The proof of the indicated generalization is similar to the proof of Theorem 1 and we only state the theorem.

**THEOREM 2.** Let the function \( f(x) \) satisfy the following conditions:

1) \( \int_{0}^{\infty} |f(x)| dx < \infty \);

2) the integral

\[
\int_{0}^{\infty} I(t) P_{n+1}^{m}(t) (t^2 - 1)^{-m/2} dt,
\]

where \( I(t) = \sqrt{\pi/2} \sum_{n=0}^{\infty} \exp(-nt) f(x) dx \), \( \Re m < \frac{1}{2} \), \( \nu \) is a complex index, converges absolutely. Then the value of the integral (10) coincides with the special case of the K-transform (3) for \( \nu = 1 \) of the function \( \phi_{m}(x) = x^{m-\frac{1}{2}} f(x) \).

Using Theorems 1–2, almost all of the integrals obtained in [1] can be generalized. Writing out the corresponding relations, we omit the details of the calculations and, furthermore, we apply Theorem 2 since it gives more general formulas.

1) \( f(x) = x^{m-k} \exp(-ax^2) \). Then,

\[
\int_{0}^{\infty} \exp\left(\frac{R^2-1}{8a}\right) D_{\nu+1}^{m}(R) P_{n+1}^{m}(R) (R^2 - 1)^{-m/2} dR = \frac{2^{\nu+m} \Gamma\left(1+\nu+m\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} W_{\nu,\nu}(1/(4a)),
\]

where \( D_{\nu}(x) \) is the parabolic cylinder function and \( W_{\nu,\nu}(z) \) is the Whittaker function, \( \Re a > 0, \Re(\mu - m/2) < 1/2 - |\Re \nu|/2, \Re m < 1/2 \).

2) \( f(x) = x^{\nu+m} J_{\nu}(ax) \), where \( J_{\nu}(ax) \) is the Bessel function. Then,

\[
\int_{0}^{\infty} (R^2 + a^2)^{\nu+m} P_{n+1}^{m}(R) (R^2 + a^2)^{\nu+m} P_{n+1}^{m}(R) (R^2 - 1)^{-m/2} dR = \frac{2^{\nu+m} a^{\nu+m} \Gamma\left(1+\nu+m\right)}{\Gamma\left(\nu+1\right) \Gamma\left(\nu+\nu+m\right)} \cdot \sum_{n} \binom{\nu+m}{n} a^{n-m} \cdot \frac{1}{2} \cdot F_{1}\left(\frac{\mu+\nu+m}{2}+1; \frac{\mu+\nu+m}{2}+1; \mu+\nu+m; a^2\right),
\]

where \( a > 0, \Re(\mu - m/2) < 1/2 - |\Re \nu|/2, \Re m < 1/2 \).