THE RATIONAL APPROXIMATION OF CONVEX
FUNCTIONS OF THE CLASS $\text{Lip} \, \alpha$

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It is proved that if a function $f(x)$ is convex on $[a, b]$ and $f \in \text{Lip}_K(f) \, \alpha$, $0 < \alpha < 1$, then the least uniform deviation of this function from rational functions of degree not higher than $n$ does not exceed $C(\alpha, \nu) (b - a) \alpha^{-1} K(f) n^{-\alpha} \ln n \ldots \ln n$ ($\nu$ is a natural number; $C(\alpha, \nu)$ depends only on $\alpha$ and $\nu$; $K(f)$ is a Lipschitz constant; and $n \geq n(\nu) = \min \{n : \ln \ldots \ln n \geq 1\}$).

INTRODUCTION

Let $R_n[f; [a, b]]$ be the least uniform deviation of a continuous function $f(x)$, $x \in [a, b]$, $-\infty < a < b < +\infty$, from rational functions of degree not higher than $n$. Suppose that $f(x)$, $x \in [0, 1]$, is convex and satisfies a Lipschitz-Hölder condition of order $\alpha > 0$ with some constant $K(f)$, $f \in \text{Lip}_K(f) \, \alpha$, i.e., $|f(x') - f(x)| \leq K(f)|x' - x|^\alpha$ for all $x', x'' \in [0, 1]$. For $R_n[f; [0, 1]]$ the following estimates are known:

(a) if $\alpha = 1$, then $R_n[f; [0, 1]] \leq C_1 K(f) n^{-2} \ln n$ (Szüsz and Turán [1], pp. 495-502); $R_n[f; [0, 1]] \leq C_2 K(n^{-2} \ln n)$ (Freud [2]); $R_n[f; [0, 1]] \leq C_1 K(f) n^{-2} \ln n \ldots \ln n$ (Popov [3]).

(b) for $0 < \alpha \leq 1$, $R_n[f; [0, 1]] \leq C_1(\alpha) K(f) (\ln n^2/n)^{1+\alpha}$ (Abdugapparov [4]); $R_n[f; [0, 1]] \leq C_1(\alpha) K(f) n^{-2} \ln n^2$ (Bulanov [5]); $R_n[f; [0, 1]] \leq C_1(\alpha) K(f) n^{-2} \ln n$ (Abdugapparov [6]).

Here $C_1, C_2, C(\nu), C(\alpha), C(f, \alpha), C_2(\alpha)$ do not depend on $n$.

On the other hand, there exists a convex function of the class Lip 1, such that $R_n[f; [0, 1]] \geq C n^{-2}$, where $C > 0$ and does not depend on $n$ (Freud [7]). Thus, the best estimate for $R_n[f]$ in the case $f \in \text{Lip}_1$, $0 < \alpha < 1$, is included between $C_1 \cdot n^{-2}$ and $C_2 n^{-2} \ln \ldots \ln n$, $C_1, C_2 > 0$, $n = 2, 3, \ldots$.

The main result of this paper is the following theorem:

THEOREM. For each function $f(x)$ that is convex and satisfies the condition $\text{Lip}_K(f) \, \alpha$, $0 < \alpha < 1$, on some segment $[a, b]$, for any natural number $\nu$ and for all $n \geq n(\nu)$

$$R_n[f; [a, b]] \leq C(\alpha, \nu) (b - a)^{\frac{\alpha}{\nu}} K(f) n^{-\alpha} \ln \ldots \ln n,$$

where $n(\nu)$ is the smallest natural number $n$ such that $\ln \ldots \ln n \geq 1$, and $C(\alpha, \nu)$ depends only on $\alpha$, $\nu$ and $0 < C(\alpha, \nu) < \infty$.

First of all, note that it suffices to prove this result for functions $f(x)$, $x \in [0, 1]$, which are convex upward, nondecreasing, continuously differentiable, belonging to the class $\text{Lip}_K(f) \, \alpha$, and equal to 0 for $x = 0$ and 1 for $x = 1$ (see [5], §1). Precisely this case will be considered. We shall prove the theorem for this case in four steps, which correspond to the four lemmas presented below.

1. LEMMA 1. Let $0 < \alpha \leq 1$; $N_0$ the smallest natural number such that $N_0 \cdot \ln^2 N_0 \geq 288 \cdot 10^2 \cdot \alpha^2$, $N \geq N_0$ a natural number; $\lambda = N^{2/\alpha}$; and $q$ a natural number such that


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Suppose further that the function \( f(x) \) is convex upward, nondecreasing, continuously differentiable, and that \( f(0) = 0 \) and \( f(1) = 1 \). Then there exist a natural number \( S \), a finite increasing sequence \( \{t_i\}_{i=0}^{S} \subset [0,1] \), a finite increasing sequence \( \{\xi_k\}_{k=0}^{S} \subset [0,1] \) containing it, and a finite sequence of natural numbers \( \{\nu_i\}_{i=0}^{S} \), having the following properties:

a) \( S \leq \frac{16 \cdot 10^3 \cdot 2^2 \cdot N \cdot \ln^2 N}{\varepsilon^2} \);

b) \( t_0 < \lambda = t_1 < t_2 < \ldots < t_{S+1} = 1 \), where \( t_i = \lambda e^{t_{i-1}}, 1 \leq i \leq q - 1 \), \( t_q = \frac{1}{2} (1 + t_{q-1}) \);

c) \( t_i - t_{i-1} \leq 2 \varepsilon (t_i - t_{i-1}), i = 1, 2, \ldots, q \);

d) \( \nu_i = \left[ m \sqrt{f(t_i) - f(t_{i-1})} + 1 \right] \), \( i = 1, 2, \ldots, q \);

e) \( \lambda_0 < \lambda = \lambda_1 < \lambda_2 < \ldots < \lambda_{S+1} = 1 \);

f) \( \xi_{k+1} - \xi_k > t(2N^{-\frac{3}{2}} \varepsilon), k = 0, 1, \ldots, S \);

g) \( \nu = \nu_i \leq \frac{1}{\varepsilon_i} (t_{i+1} - t_i) \), \( f(\xi_k) - f(\xi_{k+1}) \leq \frac{1}{\varepsilon_i} \left( \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} \right) \).

This lemma is completely analogous to an assertion of Bulanov stated and proved in [5] (pp. 480-483). The difference in the formulation is that in Lemma 1 specific values of \( m \) and \( \lambda \) are used (Bulanov imposed only the following restrictions on these parameters: \( m \geq 6, \exp(-m^2) \leq \lambda < 1 \); \( f(x) \) is used in place of \( \varphi(x) \); and conclusion d) replaced conclusion 3 of Bulanov's assertion. The proof of Lemma 1 is completely analogous to that of Bulanov's assertion; the only changes are in those calculations that arise in connection with replacing Bulanov's conclusion 3 by our conclusion d).

2. In what follows it will be convenient to use the following notation: \( L_\nu(a) = a \) for \( \nu = 0 \) and any real number \( a \), and for \( \nu > 1 \) \( L_\nu(a) = \ln \ldots \ln a \), where \( a \) is a real number such that \( L_{\nu-1}(a) \geq 0 \). Let \( \tau \) and \( \nu \) be two nonnegative integers, \( \gamma = \max\{\tau, \nu\} \), and \( n(\tau, \nu) \) the smallest natural number \( n \) such that \( L_{\nu+1}(n) > 1 \).

**Lemma 2.** Suppose that \( \alpha, N, N_f, f(x) \) and that the points \( \{t_i\}, \{\xi_k\} \) satisfy Lemma 1. If \( f \in \text{Lip}_K(f) \), then for \( n = \lceil 1000 \cdot \varepsilon^2 \ln N \rceil \), any natural number \( \nu \), and all \( N \geq N(\alpha, \nu) \), the following inequalities are simultaneously valid for all \( I_k = [\xi_k, \xi_{k+1}], k = 0, 1, \ldots, S \):

\[
R_{\nu+1} \left[ f; I_k \right] \leq C_1(\alpha, \nu) K(f) N^{-\alpha} \ln N \cdot L_{\nu+1}(n),
\]

where \( N(\alpha, \nu) \) and \( C_1(\alpha, \nu) \) depend only on \( \alpha \) and \( \nu \) and \( 1 < N(\alpha, \nu), C_1(\alpha, \nu) \leq \infty \).

**Proof.** We signify those segments \( I_k, k \geq 1 \), for which \( f(\xi_{k+1}) - f(\xi_k) > N^{-\alpha} \) by one prime. We signify those \( I_k, k \geq 1 \), for which \( f(\xi_{k+1}) - f(\xi_k) \leq N^{-\alpha} \) by two primes. The segment \( I_0 \) will be considered separately.

2a) Let us bound \( R_{\nu+1} \left[ f; I_k \right] \) from above. Let \( \varphi(x) = f(x) - f'(\xi_{k+1}) x, x \in I_k \). The function \( \varphi(x) \) is convex upward, nondecreasing, continuously differentiable, and the total variation of \( \varphi'(x) \) on \( I_k \) is equal to \( f'(\xi_k) - f'(\xi_{k+1}) \). We need the following theorem:

**Theorem (Popov [3]).** Suppose that a function \( f(x) \) on \([a,b]\) has a first derivative with finite total variation \( V^b_a(f') \). Then for each natural number \( \nu \) there exists a finite positive number \( C(\nu) > 1 \), depending only on \( \nu \) (but not on \( f \) or \([a,b]\)), such that

\[
R_{\nu} \left[ f; [a,b] \right] \leq C(\nu) V^b_a(f') (b - a) n^{-\alpha} L_{\nu}(n)
\]

for any \( n \geq n(0, \nu - 1) = \min \{n: L_{\nu}(n) > 1\} \).

Applying this theorem to \( \varphi(x) \) on \( I_k \), for any

*\([a]\) denotes the largest integer not exceeding \( a \).*