THE RATIONAL APPROXIMATION OF CONVEX
FUNCTIONS OF THE CLASS Lip $\alpha$

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It is proved that if a function $f(x)$ is convex on $[a, b]$ and $f \in \text{Lip}_K(f) \alpha$, $0 < \alpha < 1$, then the least uniform deviation of this function from rational functions of degree not higher than $n$ does not exceed $C(\alpha, \nu) (b - a)^{\nu K(f)} n^{\nu - 1} \ln \ldots \ln n$ ($\nu$ is a natural number; $C(\alpha, \nu)$ depends only on $\alpha$ and $\nu$; $K(f)$ is a Lipschitz constant; and $n \geq n(\alpha) = \min \{n : \ln \ldots \ln n \geq 1\}$).

INTRODUCTION

Let $R_n[f; [a, b]]$ be the least uniform deviation of a continuous function $f(x)$, $x \in [a, b]$, $-\infty < a < b < +\infty$, from rational functions of degree not higher than $n$. Suppose that $f(x)$, $x \in [0, 1]$, is convex and satisfies a Lipschitz-Hölder condition of order $\alpha > 0$ with some constant $K(f)$, $f \in \text{Lip}_K(f) \alpha$, i.e., $|f(x') - f(x)| \leq K(f)|x' - x|^\alpha$ for all $x', x'' \in [0, 1]$. For $R_n[f; [0, 1]]$ the following estimates are known:

(a) if $\alpha = 1$, then $R_n[f; [0, 1]] \leq C_1 K(f) n^{-2} \ln^4 n$ (Szürsz and Turan [1], pp. 495-502); $R_n[f; [0, 1]] \leq C_2 K(f) n^{-2} \ln^5 n$ (Freud [2]); $R_n[f; [0, 1]] \leq C(\nu) K(\nu) (\ln 2n/n)^{\nu - 1}$ (Popov [3]).

(b) for $0 < \alpha \leq 1$ $R_n[f; [0, 1]] \leq C_1(\alpha) K(\alpha) (\ln^5 n/n)^{\nu - 1}$ (Abdugapparov [4]); $R_n[f; [0, 1]] \leq C(\alpha) K(\nu) n^{-2} \ln^{5 - 2 \alpha} n$ (Bulanov [5]); $R_n[f; [0, 1]] \leq C_2(\alpha) K(\nu) n^{-2} \ln^{1 - 2 \alpha} n$ (Abdugapparov [6]).

Here $C_1, C_2, C(\nu), C_1(\alpha), C(f, \alpha), C_2(\alpha)$ do not depend on $n$.

On the other hand, there exists a convex function of the class Lip 1, such that $R_n[f; [0, 1]] \geq C n^{-2}$, where $C > 0$ and does not depend on $n$ (Freud [7]). Thus, the best estimate for $R_n[f]$ in the case $f \in \text{Lip} \alpha$, $0 < \alpha < 1$, is included between $C_1 n^{-2}$ and $C_2 n^{-2} \ln^{1 - 2 \alpha} n$, $C_1, C_2 > 0$, $n = 2, 3, \ldots$.

The main result of this paper is the following theorem:

THEOREM. For each function $f(x)$ that is convex and satisfies the condition $\text{Lip}_K(f) \alpha$, $0 < \alpha < 1$, on some segment $[a, b]$, for any natural number $\nu$ and for all $n \geq n(\nu)$

$$R_n[f; [a, b]] \leq C(\alpha, \nu) (b - a)^{\nu K(f)} n^{\nu - 1} \ln \ldots \ln n,$$

where $n(\nu)$ is the smallest natural number $n$ such that $\ln \ldots \ln n \geq 1$, and $C(\alpha, \nu)$ depends only on $\alpha$, $\nu$ and $0 < C(\alpha, \nu) < \infty$.

First of all, note that it suffices to prove this result for functions $f(x)$, $x \in [0, 1]$, which are convex upward, nondecreasing, continuously differentiable, belonging to the class $\text{Lip}_K(f) \alpha$, and equal to 0 for $x = 0$ and 1 for $x = 1$ (see [5], §2). Precisely this case will be considered. We shall prove the theorem for this case in four steps, which correspond to the four lemmas presented below.

1. LEMMA 1. Let $0 < \alpha \leq 1$; $N_0$ the smallest natural number such that $N_0^{-2} \ln N_0 \geq 288.10^2 \cdot \alpha^{-2}$, $N \geq N_0$ a natural number; $\lambda = N^{-2/\alpha}$, and $q$ a natural number such that $\lambda^{-2} q^{2 \lambda} \leq 1$. Then $q < 1$.

Suppose further that the function \( f(x) \) is convex upward, nondecreasing, continuously differentiable, and that \( f(0) = 0 \) and \( f(1) = 1 \). Then there exist a natural number \( N \), a finite increasing sequence \( \{t_i\}_{i=1}^N \subset [0,1] \), containing it, and a finite sequence of natural numbers \( \{\nu_i\}_{i=0}^N \), having the following properties:

a) \( N \leq 2\varepsilon N \sqrt{\ln N} \);

b) \( 0 = t_0 \leq t_1 \leq \cdots \leq t_{q+1} = 1 \), where \( t_i = \lambda t_{i-1}, 1 \leq i \leq q - 1, \ t_q = \frac{1}{2} (1 + t_{q-1}) \);

c) \( v_i = [m/\sqrt{f(t_i) - f(t_{i+1}) + 1}]^* \); \( i = 1, 2, \ldots, q \);

d) \( 0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{q+1} = 1 \);

e) \( \xi_k - \xi_{k+1} \geq 1/2N^{-2/3}, \ k = 0, 1, \ldots, S \);

Then the following two inequalities hold simultaneously:

\[
\xi_{k+1} - \xi_k \leq \frac{1}{v_i} (t_{i+1} - t_i), \quad f'(\xi_k) - f'(\xi_{k+1}) \leq \frac{1}{v_i} \frac{f(t_i) - f(t_{i+1})}{t_i - t_{i+1}}.
\]

This lemma is completely analogous to an assertion of Bulanov stated and proved in [5] (pp. 480-483). The difference in the formulation is that in Lemma 1 specific values of \( m \) and \( \lambda \) are used (Bulanov imposed only the following restrictions on these parameters: \( m \geq 6, \ exp(-m^2) \leq \lambda < 1 \); \( f(x) \) is used in place of \( \psi(x) \); and conclusion d) replaced conclusion 3 of Bulanov’s assertion. The proof of Lemma 1 is completely analogous to that of Bulanov’s assertion; the only changes are in those calculations that arise in connection with replacing Bulanov’s conclusion 3 by our conclusion d).

2. In what follows it will be convenient to use the following notation: \( L_\nu(a) = a \) for \( \nu = 0 \) and any real number \( a \), and for \( \nu > 1 \) \( L_\nu(a) = \frac{a}{\ln \cdots \ln a} \), where \( a \) is a real number such that \( L_{\nu-1}(a) > 0 \). Let \( \tau \) and \( \nu \) be two nonnegative integers, \( \gamma = \max \{\tau, \nu\} \), and \( n(\tau, \nu) \) the smallest natural number \( n \) such that \( L_\gamma(n) > 1 \).

**Lemma 2.** Suppose that \( \alpha, N_0, N, f(x) \) and that the points \( \{t_i\}, \{\xi_k\} \) satisfy Lemma 1. If \( f \in \text{Lip}_K(f) \), then for \( n = [1000\ln^2 N] \), any natural number \( \nu \), and all \( N > N(\alpha, \nu) \), the following inequalities are simultaneously valid for all \( I_k = [\xi_k, \xi_{k+1}], k = 0, 1, \ldots, S \):

\[
R_n/\|f\| \leq C_1(\alpha, \nu) K(f) N^{-2} \ln N L_{\nu+1}(N),
\]

where \( N(\alpha, \nu) \) and \( C_1(\alpha, \nu) \) depend only on \( \alpha \) and \( \nu \) and \( 1 < N(\alpha, \nu) \), \( C_1(\alpha, \nu) < \infty \).

**Proof.** We signify those segments \( I_k, k \geq 1 \), for which \( f(\xi_{k+1}) - f(\xi_k) \geq N^{-2} \) by one prime. We signify those \( I_k, k \geq 1 \), for which \( f(\xi_{k+1}) - f(\xi_k) \leq N^{-2} \) by two primes. The segment \( I_0 \) will be considered separately.

2a) Let us bound \( R_n/\|f\| \) from above. Let \( \varphi(x) = f(x) - f'(\xi_{k+1}) x, x \in I_k \). The function \( \varphi(x) \) is convex upward, nondecreasing, continuously differentiable, and the total variation of \( \varphi'(x) \) on \( I_k \) is equal to \( f'(\xi_k) - f'(\xi_{k+1}) \). We need the following theorem:

**Theorem (Popov [3]).** Suppose that a function \( f(x) \) on \( [a, b] \) has a first derivative with finite total variation \( \int_a^b |f'(x)| \). Then for each natural number \( \nu \) there exists a finite positive number \( C(\nu) > 1 \), depending only on \( \nu \) (but not on \( n, f \), or \( [a, b] \)), such that

\[
R_n/\|f\| \leq C(\nu) V_\nu^b(f')(b-a) n^{-2} L_{\nu}(n)
\]

for any \( n \geq n(0, \nu - 1) \) = \( \min \{n: L_\nu(n) > 1\} \).

Applying this theorem to \( \varphi(x) \) on \( I_k \), for any

*\([a]\) denotes the largest integer not exceeding a.*