THE RATIONAL APPROXIMATION OF CONVEX
FUNCTIONS OF THE CLASS Lip \( \alpha \)

A. Khatamov

UDC 517.51

It is proved that if a function \( f(x) \) is convex on \([a, b]\) and \( f \in \text{Lip}_K(f) \alpha, 0 < \alpha < 1 \), then the least uniform deviation of this function from rational functions of degree not higher than \( n \)
does not exceed \( C(\alpha, \nu)(b - a)\nu K(f)n^{-\alpha - 1 + \nu} \ln n \) (\( \nu \) is a natural number; \( C(\alpha, \nu) \) depends only on \( \alpha \) and \( \nu \); \( K(f) \) is a Lipschitz constant; and \( n \geq n(\nu) = \min \{ n : \ln \ln n \geq 1 \} \).

INTRODUCTION

Let \( R_n[f; [a, b]] \) be the least uniform deviation of a continuous function \( f(x), x \in [a, b], -\infty < a < b < +\infty \), from rational functions of degree not higher than \( n \). Suppose that \( f(x), x \in [0, 1] \), is convex and satisfies a Lipschitz-Hölder condition of order \( \alpha > 0 \) with some constant \( K(f) \), \( f \in \text{Lip}_K(f)\alpha \), i.e., \( |f(x') - f(x)| \leq K(f)|x' - x|^\alpha \) for all \( x', x'' \in [0, 1] \). For \( R_n[f; [0, 1]] \) the following estimates are known:

(a) if \( \alpha = 1 \), then \( R_n[f; [0, 1]] \leq C_1K(f)n^{-2}\ln^3 n \) (Szép and Turán [1], pp. 495-502); \( R_n[f; [0, 1]] \leq C_2K(f)n^{-2}\ln\ln n \) (Popov [3]).

(b) for \( 0 < \alpha \leq 1 \), \( R_n[f; [0, 1]] \leq C_1(\alpha)K(f)(\ln^5 n/\ln n)^{1+\alpha} \) (Abdugapparov [4]); \( R_n[f; [0, 1]] \leq C(f, \alpha)n^{-2}\ln^5 n \) (Bulanov [5]); \( R_n[f; [0, 1]] \leq C_2(\alpha)K(f)n^{-2}\ln^3 n \) (Abdugapparov [6]).

Here \( C_1, C_2, C(\nu), C_1(\alpha), C(f, \alpha), C_2(\alpha) \) do not depend on \( n \).

On the other hand, there exists a convex function of the class \( \text{Lip} 1 \), such that \( R_n[f; [0, 1]] \leq Cn^{-2} \), where \( C > 0 \) and does not depend on \( n \) (Freud [7]). Thus, the best estimate for \( R_n[f] \) in the case \( f \in \text{Lip} \alpha, 0 < \alpha < 1 \), is included between \( C_1n^{-2} \) and \( C_2n^{-2}\ln^3 - 0 \) in \( n, C_1, C_2 > 0, n = 2, 3, \ldots \).

The main result of this paper is the following theorem:

THEOREM. For each function \( f(x) \) that is convex and satisfies the condition \( \text{Lip}_K(f)\alpha, 0 < \alpha < 1 \), on some segment \( [a, b], \) for any natural number \( \nu \) and for all \( n \geq n(\nu) \)

\[ R_n[f; [a, b]] \leq C(\alpha, \nu)(b - a)\nu K(f)n^{-\alpha - 1 + \nu} \ln \ln n, \]

where \( n(\nu) \) is the smallest natural number \( n \) such that \( \ln \ln n \geq 1 \), and \( C(\alpha, \nu) \) depends only on \( \alpha, \nu \) and \( 0 < C(\alpha, \nu) < \infty \).

First of all, note that it suffices to prove this result for functions \( f(x), x \in [0, 1] \), which are convex upward, nondecreasing, continuously differentiable, belonging to the class \( \text{Lip}_K(f)\alpha \), and equal to 0 for \( x = 0 \) and 1 for \( x = 1 \) (see [5], §1). Precisely this case will be considered. We shall prove the theorem for this case in four steps, which correspond to the four lemmas presented below.

1. LEMMA 1. Let \( 0 < \alpha \leq 1; N_0 \) the smallest natural number such that \( N_0^{-2}N_0 > 288\cdot 10^9 \cdot \alpha^2 \), \( N \geq N_0 \) a natural number; \( \lambda = N^{-2/\alpha} \); and \( q \) a natural number such that


©1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $1.00.

1092
Suppose further that the function \( f(x) \) is convex upward, nondecreasing, continuously differentiable, and that \( f(0) = 0 \) and \( f(1) = 1 \). Then there exist a natural number \( S \), a finite increasing sequence \( \{ t_i \}_{i=0}^S \subset [0,1] \), a finite increasing sequence \( \{ \xi_k \}_{k=0}^S \subset [0,1] \) containing it, and a finite sequence of natural numbers \( \{ n_i \}_{i=0}^S \) having the following properties:

a) \( S \leq 16^{-2} \ln^{2} N \ln^{-2} N \);

b) \( 0 = t_0 \leq \xi_0 \leq t_1 \leq \xi_1 \leq \cdots \leq t_{S+1} = 1 \);

c) \( \xi_i \leq t_{i+1} \leq t_i \), \( i = 0, 1, \ldots, S \);

d) \( \xi_{S+1} = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_S = 1 \);

e) \( \xi_{S+1} - \xi_k \geq \frac{1}{2} \), \( k = 0, 1, \ldots, S \);

then the following two inequalities hold simultaneously:

\[
\frac{f' (\xi_{S+1}) - f' (\xi_k)}{\xi_S - \xi_k} \leq \frac{1}{v} \frac{f (t_i) - f (t_{i+1})}{t_i - t_{i+1}}.
\]

This lemma is completely analogous to an assertion of Bulanov stated and proved in [5] (pp. 480-483). The difference in the formulation is that in Lemma 1 specific values of \( m \) and \( \lambda \) are used (Bulanov imposed only the following restrictions on these parameters: \( m \geq 6 \), \( \exp (-m^2) \leq \lambda < 1 \); \( f(x) \) is used in place of \( \varphi(x) \); and conclusion d) replaced conclusion 3 of Bulanov's assertion. The proof of Lemma 1 is completely analogous to that of Bulanov's assertion; the only changes are in those calculations that arise in connection with replacing Bulanov's conclusion 3 by our conclusion d).

2. In what follows it will be convenient to use the following notation: \( L_0 (a) = a \) for \( \nu = 0 \) and any real number \( a \), and for \( \nu > 1 \) \( L_\nu (a) = \ln \cdots \ln a \), where \( a \) is a real number such that \( L_{\nu-1} (a) > 0 \). Let \( \tau \) and \( \nu \) be two nonnegative integers, \( \gamma = \max (\tau, \nu) \), and \( n(\tau, \nu) \) the smallest natural number \( n \) such that \( L_{\gamma+1} (n) > 1 \).

\[\text{LEMMA 2.} \quad \text{Suppose that } \alpha, N_0, N, f(x) \text{ and that the points } \{ t_i \}, \{ \xi_k \} \text{ satisfy Lemma 1. If } f \in \text{Lip}_K(f), \text{ then for } n = [1000 \gamma^{-2} \ln^2 N], \text{ any natural number } \nu, \text{ and all } N \geq N(\alpha, \nu), \text{ the following inequalities are simultaneously valid for all } I_k = [\xi_k, \xi_{k+1}], k = 0, 1, \ldots, S:
\]

\[
R_{\nu+1} [f; I_k] \leq C_1 (\alpha, \nu) K (f) N^{-2} \ln N \cdot L_{\gamma+1} (n),
\]

where \( N(\alpha, \nu) \) and \( C_1 (\alpha, \nu) \) depend only on \( \alpha \) and \( \nu \) and \( 1 < N(\alpha, \nu), C_1 (\alpha, \nu) < \infty \).

\[\text{Proof.} \quad \text{We signify those segments } I_k, k \geq 1, \text{ for which } f (\xi_{S+1}) - f (\xi_k) \geq N^{-2} \text{ by one prime. We signify those } I_k, k \geq 1, \text{ for which } f (\xi_{S+1}) - f (\xi_k) \leq N^{-2} \text{ by two primes. The segment } I_0 \text{ will be considered separately.}
\]

\[2a) \quad \text{Let us bound } R_{\nu+1} [f; I_0'] \text{ from above. Let } \varphi (x) = f (x) - f (\xi_{S+1}), \ x \in I_0'. \text{ The function } \varphi (x) \text{ is convex upward, nondecreasing, continuously differentiable, and the total variation of } \varphi' (x) \text{ on } I_k \text{ is equal to } \varphi' (\xi_k) - \varphi' (\xi_{S+1}). \text{ We need the following theorem:}
\]

\[\text{THEOREM (Popov [3]).} \quad \text{Suppose that a function } f(x) \text{ on } [a, b] \text{ has a first derivative with finite total variation } V_\delta^b (f'). \text{ Then for each natural number } \nu \text{ there exists a finite positive number } C(\nu) > 1, \text{ depending only on } \nu \text{ (but not on } n, \nu, \text{ or } [a, b]), \text{ such that}
\]

\[
R_n [f; [a, b]] \leq C (\nu) V_\delta^b (f') (b - a) n^{-2} L_\nu (n)
\]

for any \( n \geq n(0, \nu - 1) = \min \{ n: L_\nu (n) > 1 \} \).

Applying this theorem to } \varphi(x) \text{ on } I_0', \text{ for any

\[*[a] \text{ denotes the largest integer not exceeding } a.\]