RATIONAL POINTS OF ALGEBRAIC CURVES

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It is proved that the number of k-points of certain curves of genus \( g > 1 \) is finite.

The purpose of this note is to give a simple proof of the following theorem:

**Theorem.** Let \( k \) be a regular extension of a field with characteristic zero and \( \mathcal{C} \) a curve defined over \( k \). If, over some finite extension of \( k \), \( \mathcal{C} \) admits a mapping onto an elliptic curve \( \mathcal{E} \), whose smallest field of definition is not algebraic, then only a finite number of k-points lie on \( \mathcal{C} \).

Since not every curve can be mapped onto a curve of genus 1, this theorem overlaps a well-known result of Manin [1]. We shall see the appropriateness in proving it – the reasoning in certain cases can even be extended to the "numerical" variant of Mordell's conjecture. At the end of this note, this will be shown in separate examples.

Let us make some preliminary remarks.

Let \( K \) be an algebraic number field of degree \( n \) over the field of rational numbers, and let \( \mathcal{E} \) be an elliptic curve defined over \( K \). We denote a linear combination \( \sum_{j=1}^{m_j} m_j \mathcal{P}_j \) of \( K \)-points on \( \mathcal{E} \) by \( \{X(m), Y(m)\} \).

In [2] a correspondence was made between \( X(m), Y(m) \) and \( X(m), Y(m), Z(m) \), such that \( x(m) = X(m)/Z(m) \), \( y(m) = Y(m)/Z(m) \), and for each \( i \in \{1, 2, \ldots, n\} \) the number \( \mathcal{S}_i = \max\{\frac{|X^{(i)}(m)|}{|Y^{(i)}(m)|}, \frac{|Y^{(i)}(m)|}{|Z^{(i)}(m)|} \} \) has approximate quadraticity with respect to the group law of addition of points. Since the height \( H \) of \( \sum_{j=1}^{m_j} m_j \mathcal{P}_j \) can be represented in the form \( |N^{-1}(X^{(m)}(m), Y^{(m)}(m), Z^{(m)}(m))| \prod_{i=1}^{n} \mathcal{S}_i \), it follows that \( |N(X^{(m)}(m), Y^{(m)}(m), Z^{(m)}(m))| \) also has approximate quadraticity with respect to the group law. In studying the distribution of \( K \)-rational points on \( \mathcal{E} \), it is also appropriate to decompose \( |N(X^{(m)}(m), Y^{(m)}(m), Z^{(m)}(m))| \) into \( n \) components with respect to the conjugate fields \( K^{(i)} \). To do so we proceed as follows.

Let \( D \) be the nonzero discriminant of \( \mathcal{E} \), and let \( D = \prod_{p} p^r_p \) be some prime factorization of \( D \) in \( K \), in which all the divisors in \( K \) are principal and \( d_{[i]} = (X^{(i)}_{[i]}, Y^{(i)}_{[i]}, Z^{(i)}_{[i]}) \), \( d_{[i,s]} = (X^{(i)}_{[i,s]}, Y^{(i)}_{[i,s]}, Z^{(i)}_{[i,s]})/(X^{(i)}_{[i]}, Y^{(i)}_{[i]}, Z^{(i)}_{[i]}) \), where \( X_{[i]}, \ldots, Z_{[i,s]} \) are defined as in [2], and \( d_{[i], d_{[i,s]} \in K} \). Then, according to [2, 3], we have

\[
(X^{(m)}(m), Y^{(m)}(m), Z^{(m)}(m)) = (X^{(i)}(m), Y^{(i)}(m), Z^{(i)}(m)) \prod_{j=1}^{r_p} d_{[j]}^{a_{[j]}^{(m)}} \prod_{s=1}^{s_p} d_{[i,s]}^{a_{[i,s]}^{(m)}},
\]

where for \( D \neq 0 \) only the numbers \( p_t \) can be divisors of \( (X^{(i)}(m), Y^{(i)}(m), Z^{(i)}(m)) \). If \( \nu_{p_t}(q) = p_t \) is the exponent of \( q \) and \( \nu_{p_t}(X^{(m)}(m), Y^{(m)}(m), Z^{(m)}(m)) = \nu_{p_t}(X^{(i)}(m), Y^{(i)}(m), Z^{(i)}(m)) = \prod_{j=1}^{r_p} p_t^{a_{[j]}^{(m)}}. \) In this case it can be shown by the method outlined in [2] that

\[
|((X^{(i)}(m), Y^{(i)}(m)), Z^{(i)}(m))| = c_{(m), 1} \prod_{j=1}^{r_p} a_{[j]}^{m_{[j]}^{(m)}}, a_{[j]} = a_{[j]}^{(m)}.
\]

where \( c(m), i, d_j, s \) depend only on the coefficients of \( \mathcal{G} \) and the coordinates of \( P_j \). Thus, each of the components of the Weil height \( H_i \left( \sum_{j=1}^{n} m_j P_j \right) = \mathcal{G}_i \left( \sum_{j=1}^{n} m_j P_j \right) / \prod ((X_{ij}^{(t)})^j, (Y_{ij}^{(t)})^j, (Z_{ij}^{(t)})^j) \) has approximate quadracity with respect to the group law of addition of points on \( \mathcal{G} \).

If \( K \) is an extension of the function field \( k(t) \) of degree \( n \), then, by setting \( t = t_1/t_0 \) and choosing prime factorizations of \( D_i, (X_{ij}^{(t)}, Y_{ij}^{(t)}, Z_{ij}^{(t)}) \) in \( K_1 \) such that under the substitution \( \{t, t_0\} \rightarrow \{ft_1, ft_0\} \) the relations \( p_{ij} \rightarrow p_{ij}^{(t_1, t_0)}, d_{ij} \rightarrow d_{ij}^{(t_1, t_0)} \) hold, i.e., \( p_{ij}, d_{ij}, s \) are homogeneous with respect to \( t_1, t_0 \), in a similar way we construct \( n \) components of the Weil height. Note only that in this case by the height \( H_i \left( \sum_{j=1}^{n} m_j P_j \right) \) of \( \sum_{j=1}^{n} m_j P_j \) we mean \( \deg \left( \sum_{j=1}^{n} m_j P_j \right) \) with respect to \( t_1, t_0 \).

We now turn to the proof of the theorem. We shall prove it by contradiction. Let \( k = k_0(t_1, t_0) \), where \( k_0 \) is an algebraically closed field of characteristic zero. Let \( \rho \) be a root of the equation \( g = 0 \) defined over \( k \) and whose left side is sent by some \( \beta \) into \( \mathcal{G}g \) under the substitution \( \{p, t_1, t_0\} \rightarrow \{\beta p, t_1, t_0\} \). Let \( K \) be an extension of \( k(p) \) of degree \( n > 1 \). Obviously, without loss of generality we can assume that \( \mathcal{G} \) is a projective curve defined over \( k \) with coefficients that are homogeneous with respect to \( t_1, t_0 \), has points with coordinates in \( k(p) \); and admits a mapping \( \varphi = \{z^2, \psi(x, y, z), \xi(x, y, z)\} \) onto an elliptic curve \( \mathcal{G} \). Let \( K(i), i = 1, 2, \ldots, n \), be fields conjugate to \( K \) with respect to \( k(p) \), and for \( K \rightarrow K(i) \rightarrow K(i+1) \) we denote basic points on \( \mathcal{G}(i) \) by \( Q_i, i = 1, 2, \ldots, n \). Then for any \( k(p) \)-point \( P \) of \( \mathcal{G} \) we have

\[
\begin{align*}
\psi(0)(P)/d(0) &= e(0), x^{(0)} \psi(0)(P)/d(0) = e(0), x^{(0)} \psi(0)(P)/d(0) = e(0), x^{(0)} \psi(0)(P)/d(0), \\
d(0) &= ((\psi(0)(P))^2, (\psi(0)(P))^2, (\xi(0)(P))^2), \\
D(i) &= ((X(i)^2), (Y(i)^2), (Z(i)^2)),
\end{align*}
\]

where \( e(0), i \) is a unit in \( K(i) \). Since \( d(0)(P) \) and \( D(0)(\sum_{j=1}^{n} m_j Q_j) \) are elements of \( K(i) \) that are homogeneous with respect to \( t_1, t_0 \), the units conjugate to \( e(0), i \) with respect to \( k \) satisfy the equation \( \sum_{i=0}^{N} d_{i, x} z^n = 0 \), where \( d_n = A_n/B_n \) and \( A_n, B_n \) are homogeneous polynomials of the same degree with respect to \( t_1, t_0 \). The units \( e \) are integral elements of the fields \( K(i) \), and so \( A_n \equiv 0 \mod B_n \). On the other hand, \( t_1, t_0 \) are arbitrary parameters; therefore, \( d_n = A_n/B_n \) is a number in \( k_0 \). Thus, the units \( e(0), i = e(0) \) are not functional and depend only on \( k_0 \). Furthermore, since \( n > 1 \), by hypothesis, if a point \( P \) with a sufficiently large height in comparison with the heights of \( \mathcal{G} \) lies on \( \mathcal{G}(i) \), then for any fixed values \( t_1, t_0 \)

\[
|\ln H_i(\eta(P))/H_i(\eta(P)) - \sum_{j=1}^{n} (d_j, s_j, t_j, u_j - d_j, s_j, t_j, u_j, m_j m_j/\max \{m_j\})| < \varepsilon, \quad d_j, s_j = d_j, s_j,
\]

where \( \varepsilon \) is arbitrarily small. Consequently,

\[
-\varepsilon < \sum_{j=1}^{n} (d_j, s_j, t_j, u_j - d_j, s_j, t_j, u_j, m_j m_j/\max \{m_j\}) < \varepsilon,
\]

whence, bearing in mind that for fixed \( t_1, t_0 \) \( d_j, s_j, t_j, u_j \) are fixed numbers, we deduce that

\[
\det \begin{bmatrix} d_j, s_j, t_j, u_j - d_j, s_j, t_j, u_j, m_j \end{bmatrix} = 0. \tag{1}
\]

But (1) is impossible for \( i_1 \neq i_2 \) and arbitrarily chosen \( t_1, t_0 \). The resulting contradiction proves the theorem.

In conclusion, with the help of these components of the Weil height, we consider two examples concerning the "numerical" variant of Mordell's conjecture.

The projective curve

\[
x^4 + y^4 = dz^4 \tag{2}
\]

over the field of rational numbers \( R \) admits the mappings

\[
\eta_1 = \{x/z, y^2/z^2\}, \quad \eta_2 = \{y, x^2/z^2\}, \quad \eta_3 = \{x/y, z^2/y^2\}
\]

onto the curves

\[
u^4 + v^2 = d \tag{3}
\]