A general approach is proposed to the interpolation of \( x^a \)-analytical functions of a complex variable with an arbitrary \( a \in ]-\infty, +\infty[ \). Basis \( x^a \)-analytical functions whose imaginary part is a polynomial in \( x \) and \( y \) are obtained in explicit form.

Some topics connected with the interpolation problem for \( p \)-analytical functions of a complex variable [5] and with existence and uniqueness of the solution of this problem were examined in detail in [1-4] for some particular cases of the characteristic \( p \). In this paper, we propose a general approach to interpolation of \( x^a \)-analytical functions of complex variable for an arbitrary \( a \in ]-\infty, +\infty[ \). We consider the following problem. Find a function \( W_m(z) = u_m(x, y) + iv_m(x, y) \) of a complex variable \( z = x + iy \) which is \( x^a \)-analytical in some simply connected domain \( D \) and its imaginary part is a polynomial in \( x \) and \( y \) of lowest possible degree \( n \) such that

\[
W_m(z_j) = s_j, \quad j = 1, \ldots, m, \tag{1}
\]

where \( z_j = x_j + iy_j \) are arbitrary pairwise distinct points in the complex halfplane \( x > 0; s_j = a_j + ib_j, j = 1, \ldots, m \), are given arbitrary complex-valued constants.

We first find the basis \( x^a \)-analytical functions \( \bar{W}_r(z) = u_r(x, y) + iv_r(x, y) \) whose imaginary part is a homogeneous polynomial in \( x \) and \( y \) of degree \( r \) (\( r = 0, 1, \ldots \)). Seeing that \( \bar{v}_r(x, y) \) has the form

\[
\bar{v}_r(x, y) = \sum_{l=0}^{r} a^{(r)}_{l} x^l y^{r-l},
\]

where \( a^{(r)}_{l}, l = 0, \ldots, r \), are some as yet undefined coefficients, we obtain from the conditions of \( x^a \)-analyticity

\[
a^{(r)}_{l} = 0, \quad \forall r = 0, 1, \ldots,
\]

\[
(1 + 2)(1 + 1 - \alpha) a^{(r+2)}_{l+2} + (r - l)(r - l - 1) a^{(r)}_{l} = 0, \quad l = 0, r - 2.
\]

Let us consider some different cases.

1. \( \alpha \notin \mathbb{N} \). Then all the coefficients in Eq. (3) are nonzero for any \( l \), i.e., there exists a one-parameter family of homogeneous polynomials and it is easy to show that these polynomials are representable in the form

\[
\bar{v}_r(x, y) = Q_r(x, y) = \sum_{k=0}^{[r/2]} \alpha_k^{(r)} x^{(r-k)} y^{-2k}, \quad r = 0, 1, 2, \ldots,
\]

where

\[
\alpha_k^{(r)} = \begin{cases} 
1, & k = 0 \\
\frac{\Gamma(r + 1) \Gamma(r - 1 - k)}{\Gamma(k + 1) \Gamma(\alpha - 1) \Gamma(\alpha - 3 - 2k + 1) \Gamma(\alpha - 2k + 1)}, & k = 1, [r/2].
\end{cases}
\]

Here and in what follows, \([a]\) is the whole part of the number \( a \).
Reconstructing the real part of an \( x^\alpha \)-analytical function from its imaginary part, we obtain

\[
\widetilde{u}_r(x, y) = x^{-\alpha} P_r(x, y) + C_r,
\]

where

\[
P_r(x, y) = \sum_{l=0}^{(\alpha+1/2)} \frac{r - 2k - \alpha + 1}{2k - \alpha + 1} \alpha_{2k+1} y^{-2k-1},
\]

(6)

\( C_r \) is an arbitrary constant.

Thus, up to a (real) constant multiplier, the basis \( x^\alpha \)-analytical functions in this case are given by

\[
\widetilde{W}_r(z) = \left( \frac{P_r(x, y)}{x^\alpha} + C_r \right) + iQ_r(x, y), \quad r = 0, 1, 2, \ldots,
\]

where \( P_0(x, y) = 0 \), the polynomials \( P_r(x, y), r = 1, 2, \ldots \), and \( Q_r(x, y), r = 0, 1, \ldots \), are defined by formulas (4)-(6).

The solution of our interpolation problem is now sought in the form

\[
W_m(z) = \sum_{r=0}^{n} a_r \widetilde{W}_r(z) = \left( \sum_{r=0}^{n} a_r x^{-\alpha} P_r(x, y) + C \right) + i \sum_{r=0}^{n} a_r Q_r(x, y),
\]

(7)

where

\[
C = \sum_{r=0}^{n} a_r C_r.
\]

(8)

To find the unknowns \( a_r, r = 0, \ldots, n \), and \( C \), we apply conditions (1) and obtain the following system of linear equations:

\[
\begin{align*}
Cx_i + \sum_{r=0}^{n} a_r P_r(x_i, y_j) &= \alpha_j x_i^\alpha, \\
\sum_{r=0}^{n} a_r Q_r(x_i, y_j) &= \beta_j, \quad j = 1, m.
\end{align*}
\]

(9)

To ensure that system (9) has a solution for any \( \alpha_j, \beta_j, j = 1, \ldots, n \), we should first impose the equality

\[
n = 2m - 2.
\]

(10)

Condition (10) establishes a relationship between the number of interpolation points and the least possible degree of the polynomial representing the imaginary part of the \( x^\alpha \)-analytical interpolant for given \( \alpha \).

2. \( \alpha = 2k \). If \( r \geq 2k + 1 \), then the coefficient \( (l + 1 - \alpha) \) in (3) vanishes for an appropriate \( l \). This indicates that there exists a two-parameter family of homogeneous polynomials \( \varphi_r(x, y) \): \( Q_r(1)(x, y) \) and \( Q_r(2)(x, y) \). After some manipulations, we show that for \( \alpha = 2k \) (\( k \in \mathbb{N} \)) the solution of the interpolation problem is

\[
W_m(z) = \frac{P_{n-1}(x, y)}{x^{2k-1}} + iQ_n(x, y),
\]

(11)

where

\[
P_n(x, y) = \begin{cases} 
\sum_{r=0}^{n} a_{r+1} P_{r}^{(1)}(x, y) + C x^{2k-1}, & n \leq 2k \\
\sum_{r=0}^{n} a_{r+1} P_{r}^{(1)}(x, y) + \sum_{r=2k+1}^{n} a_{r+1} P_{r+1}^{(2)}(x, y) + C x^{2k-1}, & n > 2k;
\end{cases}
\]

(12)