SIMILARITY AND THE STRUCTURE OF UNITAL MATRIX QUADRATIC TRINOMIALS WITH PAIRWISE DIFFERENT CHARACTERISTIC ROOTS

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The normal form of certain polynomial matrices is found.

In this paper we find a normal form in the class of \( \Lambda \)-equivalent matrices for a certain numeric matrix that is closely related to the \( n \times n \) polynomial matrix \( A(x) = Ex^2 + A_1x + A_2 \) with characteristic polynomial \( \Delta(x) = \det A(x) \) having only simple roots. The existence of this form solves a problem on similarity of pairs of matrices, and reveals the structure of the matrix \( A(x) \) in the case under consideration.

All of the numeric matrices and polynomials are considered over an algebraically closed field of characteristic zero.

Two polynomial matrices \( A(x) \) and \( B(x) \) are said to be semiscalar equivalent if there exist a nonsingular numeric matrix \( S \) and an invertible polynomial matrix \( Q(x) \) such that \( A(x) = SB(x)Q(x) \). It is clear that the matrices \( A(x) = E(x^2) + A_1(x) + A_2 \) and \( B(x) = E(x^2) + B_1x + B_2 \), \( \det B(x) = \Delta(x) \), are semiscalar equivalent if and only if they, or, what is the same, the pairs of matrices \((A_1,A_2)\) and \((B_1,B_2)\), are similar.

It is known ([1], p. 131) that semiscalar equivalent transformation of two matrices \( A(x) \) and \( B(x) \) leads to the triangular forms

\[
S_1 A(x) Q_1(x) = \begin{pmatrix}
1 & \cdots & 0 \\
0 & \ddots & \vdots \\
-a_1(x) & \cdots & -a_{n-1}(x) \Delta(x)
\end{pmatrix}, \\
S_2 B(x) Q_2(x) = \begin{pmatrix}
1 & \cdots & 0 \\
0 & \ddots & \vdots \\
-b_1(x) & \cdots & -b_{n-1}(x) \Delta'(x)
\end{pmatrix}
\]  

(1)

where \( \deg a_i(x) < 2n \), \( \deg b_i(x) < 2n \), \( i = 1, \ldots, n-1 \).

We consider the rows \( a(x) = [a_1(x) \ldots a_{n-1}(x)] \) and \( b(x) = [b_1(x) \ldots b_{n-1}(x)] \), where the \( a_i(x) \) and \( b_i(x) \), \( i = 1, \ldots, n-1 \), are taken from the triangular forms of (1). Let

\[
M_{a(x)}(\Delta) = \begin{pmatrix}
a_1(x) & \cdots & a_{n-1}(x) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & a_{2n} & a_{2n-1}(x) & 1
\end{pmatrix}, \\
M_{b(x)}(\Delta) = \begin{pmatrix}
b_1(x) & \cdots & b_{n-1}(x) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & b_{2n} & b_{2n-1}(x) & 1
\end{pmatrix}
\]  

(2)

be the values of these rows on the system of roots of the polynomial \( \Delta(x) \) with fixed enumeration ([1], p. 27).


**Theorem 1.** The matrices \( A(x) \) and \( B(x) \) are similar if and only if there exist a nonsingular numeric matrix \( C \) and a diagonal matrix \( D \) such that \( M_{A(x)}(\Lambda) = D M_{B(x)}(\Lambda) C. \)

**Proof.** It follows from (1) and the similarity of \( A(x) \) and \( B(x) \), i.e., \( A(x) = T B(x) T^{-1} \), where \( T \) is a nonsingular numeric matrix, that \( CS_1 A(x) Q_1(x) = S_2 B(x) Q_2(x) Q(x) \). In explicit form, we have

\[
\begin{bmatrix}
\Delta(x) & 0 \\
0 & \Delta(x)
\end{bmatrix}
= \begin{bmatrix}
c_{11} & \cdots & c_{1n} \\
0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
c_{n1} & \cdots & c_{n,1} & \cdots & c_{nn}
\end{bmatrix}
\]

(3)

If we consider the values of the matrices in both parts of (3) on the system of roots of the polynomial \( \Delta(x) \), we obtain

\[
M_{\Delta(x)}(\Delta) C_* = diag(d_1, \ldots, d_{2n}) M_{\delta(x)}(\Delta),
\]

where \( d_i = d_{nn}(c_i) \), and, by (3), \( d_{nn}(c_i) \neq 0 \), and \( C_* \) is a nonsingular matrix.

Conversely, if (4) is true, there exists a polynomial \( d_{nn}(x) \) that is of degree less than \( 2n \) and such that \( d_{nn}(c_i) = d_i \), \( i = 1, \ldots, 2n \). Since \( (d_{nn}(x), \Delta(x)) = 1 \), we can see, after we construct the equation

\[
\begin{bmatrix}
\alpha_1(x) & \ldots & \alpha_{n-1}(x)
\end{bmatrix} C_* = \begin{bmatrix}
d_{n1}(x) & \ldots & d_{n,1}(x) & \ldots & d_{nn}(x) & \Delta(x)
\end{bmatrix}
\]

(5)

and is invertible. The equation obtained from (5) with \( X = Q(x) \) implies the semiscalar equivalence of the matrices in (1). Q.E.D.

**Definition 1.** We will call the following three transformations of a numeric matrix \( M \) \( \Delta \)-equivalent transformations: permutation of rows of \( M \); multiplication of \( M \) by a diagonal numeric nonsingular matrix on the left; multiplication of the matrix \( M \) by a numeric nonsingular matrix on the right.

**Definition 2.** Two numeric matrices are said to be \( \Delta \)-equivalent if one can be transformed into the other by \( \Delta \)-equivalent transformations.

**Definition 3.** A sequence of \( l \) rows \( \alpha_{k_1}, \ldots, \alpha_{k_l} \) of a numeric matrix is said to be connected if each submatrix formed from two adjacent rows \( \alpha_{k_i}\alpha_{k_{i+1}}, i = 1, \ldots, l - 1 \), in this sequence contains a column of the form \( [1 \ 1]^T \).

We will call any two rows belonging to a single connected sequence of rows in a given matrix equivalent.

**Definition 4.** We call a numeric matrix \( H \) standard if all of its first nonzero elements in both rows and columns are ones, and all of its rows form a connected sequence.

By the first nonzero element in a row (column), as usual, we mean the leftmost (rightmost) nonzero element in a row (column).