SINGULARITIES OF SPHEROIDAL METRICS

A. P. Slivinskii

An analysis is made of the singularities of Weyl gravitational fields in the spheroidal coordinates of oblate and prolate ellipsoids of revolution. Analysis of the Kretschmann scalars calculated shows them to be directional, so the singularities have a non-Schwarzschild topology. These singularities are accompanied by the appearance of "conjugate" regions, whose physical interpretation is that the active mass of the singularity changes its sign to that opposite the sign of the mass of a test particle which enters the conjugate region. These singular objects are designated as being of types PO and OP; they have images OP* and PO*, respectively, in the conjugate regions. The images of such objects, which attract matter into the conjugate region, act as explosive sources of matter and radiation in our region of space.

INTRODUCTION

In the solution of the equations for a gravitational field with central symmetry a singularity arises at a sphere with gravitational radius rg. All the invariants are finite on this sphere, as is the determinant g constructed from the elements of metric tensor g_{ik}. The time required for a test particle to reach the singular surface is asymptotically infinite for an outside observer, but finite for a comoving observer. The force exerted on the test particle at the singular surface is infinite. Here and below, "force" means the vector difference between the forces appearing in the geodesic-deviation equation.

We turn now to a more complicated case - that of the equation for a gravitational field of axial symmetry. The Kretschmann scalar was calculated in cylindrical coordinates in [1], and the directional singularity of the scalar was demonstrated in [2] through the use of the particular solution of a static gravitational field. As \( \rho \to 0 \), this scalar diverges, while as \( z \to 0 \), it remains finite.

Spheroidal coordinates (the coordinates of an oblate ellipsoid of revolution) were used in [3] for an analysis of the singularity. Solutions of the lowest multipole orders were used without any justification.

The coordinates of a prolate ellipsoid of revolution were used in [4] to calculate the Kretschmann scalar in the radial plane through the use of a "quadrupole" solution; it was found that this scalar diverges as \( \rho \to 0 \).

Here we are also concerned with the singularities of the solutions for a static gravitational field in the spheroidal coordinates of prolate and oblate ellipsoids of revolution. We assume that we need take into account only the lowest multipole order. In this case, as we will see, the field source lies under the singular surface, as in the case of the Schwarzschild field.

1. The \( P \) Region

The vacuum equations for a static gravitational field in the axisymmetric case are [5]

\[
\Delta \phi = 0, \\
\gamma_{\rho} = \frac{1}{2} \left( \gamma_{\rho} - \frac{1}{2} \right), \quad \gamma_{\phi} = 2 \phi_{\phi} \phi_{\phi}.
\]

(1.1)

(1.2)

Here the operator \( \Delta \) is the Laplacian written in terms of cylindrical coordinates without the azimuthal part.
We will use a unit system in which we have $G = c = 1$ (c is the speed of light; and $G$ is the gravitational constant). In this unit system, time and mass have the dimensionality of length.

In the coordinates of a prolate ellipsoid of revolution [6], the linear element is

$$\text{ds}^2 = e^{2\psi} \text{d}t^2 - a^2 e^{2(1-\lambda)} (\text{d}x^2 - \text{d}y^2) - \frac{\text{d}r^2}{1 - \nu^2} - a^2 (\lambda^2 - 1) (1 - \nu^2) e^{-2\psi} \text{d}\psi^2. \quad (1.3)$$

Here $2a$ is the focal distance. The field equations are

$$\begin{align*}
\gamma_\lambda &= \frac{1}{\lambda^2 - \nu^2} \left[ \nu (\lambda^2 - 1) \psi_\lambda - \nu (1 - \nu^2) \psi_\nu - 2 \nu (\lambda^2 - 1) \psi_\lambda \psi_\lambda \right], \\
\gamma_\nu &= \frac{\lambda^2 - 1}{\lambda^2 - \nu^2} \left[ \nu (\lambda^2 - 1) \psi_\lambda - \nu (1 - \nu^2) \psi_\nu + 2 \nu (1 - \nu^2) \psi_\lambda \psi_\nu \right].
\end{align*} \quad (1.4, 1.5)$$

The general solution of (1.4) is

$$\psi = \sum_{n=0}^{\infty} b_n P_n (\nu) Q_n (\lambda). \quad (1.6)$$

Here $P_n (\nu)$ and $Q_n (\lambda)$ are Legendre polynomials of the first and second kinds, respectively. We assume that the basic contribution comes from the zeroth term, finding

$$\begin{align*}
\psi &= -b \arctan \frac{1}{\lambda}, \\
\gamma &= \frac{b^2}{2} \ln \frac{\lambda^2 - 1}{\lambda^2 - \nu^2}. \quad (1.7, 1.8)
\end{align*}$$

From the condition that we find a Newtonian potential in the case $\nu \gg 1$, we find $b = m/a$, where $m$ is the active mass of the field source. In particular, with $b = 1$ we have $\lambda = (r^* - 1)/m$, $\nu = \cos \theta$; element (1.3) becomes the Schwarzschild linear element with the natural coordinate $r^*$, the distance from the origin. However, we have

$$\lambda = \left[ \frac{r^2 + a^2 + (r^* + a^2 + 2a^2 r^2 (1 - 2 \sin^2 \theta))^{1/2}}{2a} \right]^{1/2},$$

where $r$ and $\theta$ are spherical coordinates. At the origin of our coordinate system ($r = 0$) we evidently have $\lambda = 1$, and the natural Schwarzschild coordinate is $r^* = 2m$; i.e., in this case, as in the case of a Schwarzschild field, we can consider a source under the singular surface with $\lambda = 1$.

As in the case of the Schwarzschild field, the determinant $g$ is finite; at $\lambda = 1$, we have $g = 16 m^6$.

To study the singularity, we evaluate the Kretschmann scalar $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$:

$$K = 144 \left( \frac{b^2}{a^2} \frac{e^{4(\nu - \lambda)}}{(\lambda^2 - 1)^2 (\lambda^2 - \nu^2)} \right)^{1/2} \left[ \nu (1 - \nu^2) \left( b^2 (4 \lambda^2 \nu^2 - 3 \nu^2 - \lambda^2) - 3 \nu^2 \nu^2 + 2 \lambda^2 - \lambda^2 + (\lambda^2 - \nu^2) \right) \right]^2 + \left[ \nu (1 - \nu^2) \left( b (3 \lambda^2 - 2 \lambda^2 - \lambda) - 2 \nu^2 (\lambda^2 - 1) \right) \right]^2 + \left( \lambda^2 - \nu^2 \right) \left( b (3 \lambda^2 - 2 \lambda^2 - \lambda) - 2 \nu^2 (\lambda^2 - 1) \right)^2 + \left( \lambda^2 - 1 \right)^2 \left( b (3 \lambda^2 - 2 \lambda^2 - \lambda) - 2 \nu^2 (\lambda^2 - 1) \right)^2 \right]. \quad (1.9)$$

Here $\psi$ and $\gamma$ are given by (1.7) and (1.8), respectively. The directional singularity of this scalar is obvious. In particular, with $b = 1$ and $\mu = 1$, i.e., at the symmetry axis, $K$ is finite and nonvanishing at $\lambda = 1$, as in a Schwarzschild field, but diverges for other $\mu$. For $b > 1$, points appear on the symmetry axis with a planar metric at $\lambda = b$, for we have $K = 0$ at these points. This circumstance imposes an upper limit on the allowed values of $b$: $b \leq 1$. With $b < 1$, $K$ diverges over the entire $\lambda = 1$ singular surface.