A completeness relation is derived for the set of solutions of the radial Schrödinger equation with a nuclear-Coulomb potential in the complex $\lambda$-plane for a fixed physical energy.

1. Introduction

We consider the Schrödinger equation

$$\left(\frac{d^2}{dr^2} + \frac{\lambda^2}{r^2} + \frac{1}{4} - \frac{a}{r} - V(r)\right)\psi(\lambda, \kappa, r) = 0,$$  \hspace{1cm} (1.1)

where $\kappa$ takes on the physically meaningful values ($\kappa > 0$); $a$ is a real number; and $\lambda$ is complex. We choose as $V(r)$ the generalized Yukawa potential

$$V(r) = \int_{m > 0}^{\infty} \frac{\exp(-mr)}{r} d\nu, \hspace{1cm} \int_{m}^{\infty} \frac{|z(\nu)|}{\nu} d\nu < \infty.$$  \hspace{1cm} (1.2)

We find the regular solution $\psi(\lambda, \kappa, r)$ of Eq. (1.1) by means of the boundary condition

$$\lim_{r \to 0} \psi(\lambda, \kappa, r) r^{-\lambda - \frac{1}{2}} = 1.$$ \hspace{1cm} (1.3)

The Jost solutions $f(\lambda, \pm \kappa, r)$ are found by means of the boundary conditions at infinity:

$$\lim_{r \to \infty} f(\lambda, \pm \kappa, r) \exp\left(\frac{a\pi}{4\kappa} \text{sign}(\kappa)\right) \exp\left[ \pm i \left(\kappa r - \frac{a}{2\kappa} \ln(2\kappa r)\right)\right] = 1.$$ \hspace{1cm} (1.4)

If there is no nuclear potential ($V = 0$), the free solutions are

$$\psi_0(\lambda, \kappa, r) = (2i\kappa)^{-\lambda - \frac{1}{2}} M_{\frac{\lambda}{2\kappa}}(2\kappa r); \hspace{1cm} f_0(\lambda, \pm \kappa, r) = W_{\frac{\lambda}{2\kappa}, \lambda}(\pm 2\kappa r),$$ \hspace{1cm} (1.5)

where $M_{\frac{\lambda}{2\kappa}, \lambda}$ and $W_{\frac{\lambda}{2\kappa}, \lambda}$ are Whittaker functions.

Equation (1.1) has the four solutions $\psi(\pm \lambda, \kappa, r)$, $f(\lambda, \pm \kappa, r)$. Using boundary conditions (1.3) and (1.4), we can easily find the Wronskians:

$$W[\psi(\lambda, \kappa, r), \psi(-\lambda, \kappa, r)] = -2\lambda;$$

$$W[f(\lambda, \kappa, r), f(\lambda, -\kappa, r)] = 2i\kappa \exp\left(-\frac{a\pi}{2\kappa} \text{sign}(\kappa)\right).$$ \hspace{1cm} (1.6)


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The Jost function $f(\lambda, -\kappa)$ is given by

$$f(\lambda, -\kappa) = \mathcal{W}[f(\lambda, -\kappa, r), \varphi(\lambda, \kappa, r)]. \quad (1.7)$$

As in [1], we use (1.6) and (1.7) to find the following relations ($\kappa > 0$):

$$f(\lambda, -\kappa, r) = \frac{1}{2\kappa} [f(\lambda, -\kappa) \varphi(-\lambda, \kappa, r) - f(-\lambda, -\kappa) \varphi(\lambda, \kappa, r)]; \quad (1.8)$$

$$\varphi(\lambda, \kappa, r) = \frac{e^{2\alpha}}{2\kappa^2} [f(\lambda, \kappa) f(\lambda, -\kappa, r) - f(\lambda, -\kappa) f(\lambda, \kappa, r)]. \quad (1.9)$$

It follows from (1.8) that $f(\lambda, -\kappa, r)$ is an even function with respect to $\lambda$.

The results of [2] show that $f(\lambda, -\kappa)$ is a holomorphic function in the region $\text{Re} \lambda > 0$.

In the case of a Coulomb potential, we have the following for Jost function $f_0(\lambda, -\kappa)$ [3]:

$$f_0(\lambda, -\kappa) = \frac{(2i\kappa)^{-\lambda+\frac{1}{2}}}{\Gamma(\lambda + \frac{1}{2})} \frac{\Gamma(2\kappa + 1)}{\Gamma\left(\frac{1}{2} + \frac{4\alpha}{2\kappa}\right)} . \quad (1.10)$$

Below we will need the asymptotic forms of the functions $f(\lambda, -\kappa)$ and $\varphi(\lambda, \kappa, r)$ for large $\lambda$. It was shown in [4] that under condition (1.2) the Jost functions for a nuclear-Coulomb potential behave at large $\lambda$ in the region $\text{Re} \lambda > 0$ like the Jost functions for a Coulomb potential, so we can write

$$f(\lambda, -\kappa) \sim f_0(\lambda, -\kappa) (2\pi \kappa)^{-\lambda+\frac{1}{2}} \frac{1}{2} \frac{2^{4\alpha/2\kappa}}{\Gamma\left(\frac{1}{2} + \frac{4\alpha}{2\kappa}\right)}, \quad (|\lambda| \to \infty). \quad (1.11)$$

From Eqs. (2.9) and (2.10) [4] we find two different expressions (corresponding to contours $\Gamma_0$ and $\Gamma_1$) for $\varphi(\lambda, \kappa, z)$:

$$\varphi(\lambda, \kappa, z) = \exp\left(4i\kappa\right) B_0 p_0 \frac{1}{2i} \int \frac{d \tilde{z}}{0, \Gamma_0} \frac{1}{2i} \int \exp(2i\tilde{z}) I(\tilde{z}) Y_0(\tilde{z}) \, d\tilde{z};$$

$$\varphi(\lambda, \kappa, z) = \exp\left(-4i\kappa\right) B_0 p_0 \frac{1}{2i} \int \frac{d \tilde{z}}{0, \Gamma_1} \frac{1}{2i} \int \exp(-2i\tilde{z}) I(\tilde{z}) Y_0(\tilde{z}) \, d\tilde{z}. \quad (1.12)$$

Equations (2.1a), (2.2), and (2.3) of [4] yield the following for large $\lambda$:

$$\xi_0(z) \sim \xi_{0,e}(z) \sim -\bar{\mu} - \frac{a}{2\kappa} \ln(2\kappa) + \frac{ia\pi}{4\kappa} - \bar{\mu} \ln z - \bar{\mu} \ln\left(\frac{\kappa}{2\kappa}\right) + \frac{\pi}{2} \lambda; \quad (1.13)$$

$$\xi_1(z) \sim \xi_{1,h}(z) \sim -\bar{\mu} - \frac{a}{2\kappa} \ln(2\kappa) + \frac{ia\pi}{4\kappa} + \bar{\mu} \ln z + \bar{\mu} \ln\left(\frac{\kappa}{2\kappa}\right) + \frac{\pi}{2} \lambda.$$ Here a plus sign is used with the root for contours $\Gamma_0$, and a minus sign is used for $\Gamma_1$ [4].

Substituting (1.13) into (1.12), and using the values of $B_0, B_1, p_0,$ and $p_1$ for large $\lambda$ (see Eqs. (2.2), (2.3), (2.11), and (2.3) in [4]), we find that the asymptotic form of the regular particular solution $\varphi(\lambda, \kappa, r)$ which satisfies boundary condition (1.5) of [4] does not depend on the nature of contours $\Gamma_0$ and $\Gamma_1$ and is given by

$$\varphi(\lambda, \kappa, z) \sim \frac{\Gamma\left(\lambda + \frac{1}{2} + \frac{4\alpha}{2\kappa}\right)}{\Gamma(2\kappa + 1)} (-2i\kappa)^{\lambda+\frac{1}{2}}, \quad (|\lambda| \to \infty). \quad (1.14)$$

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