FINITE ELEMENT CONSTRUCTION ON THE BASIS OF ANALYTIC
SOLUTIONS OF ELASTICITY THEORY PROBLEMS

I. P. Kréchun

A finite element in the form of a part of a circular ring loaded by a uniformly distributed radial load is constructed on the basis of the exact solution of an elasticity theory problem. This permits a system of resolving equations to be obtained by the finite element method with a three-diagonal matrix. An example is presented that illustrates the accuracy and efficiency of utilization of the elements mentioned.

The standard procedure of the finite element method is based on an approximate representation of the stress-strain state of the element, as a rule, by using a system of interpolation functions or shape functions [1, 3]. To assure the necessary accuracy of the solution such an approach requires a sufficiently compact network of elements, which results in the need for a large volume of calculations and, particularly, in operation with systems of high order algebraic equations with a significant tape width for the appropriate matrices. The difficulties mentioned can be overcome in a number of cases by applying finite elements for whose construction known exact analytic solutions of elasticity theory problems are used.

As an example, let us consider the plane problem of the stress state of an infinite wedge (half-plane) in a polar coordinate system with a circular indentation at the origin, loaded by a uniformly distributed radial load. We represent the computational domain by a system of finite elements in the form of a part of a circular ring bounded by the radii \( r_1 \), \( r_2 \) and the rays \( \theta = \pm \theta^0 \) (Fig. 1). By virtue of symmetry there are no tangential stresses on the contact surface of the elements and radial normal stresses \( \sigma_{rr} \) will be interaction forces between them. The endfaces of an element with normal \( S \) are considered stress-free and mass forces are not taken into account. Let us use the known solution of the biharmonic equation in polar coordinates in terms of the stress function [4]

\[
\varphi = A \ln r + B^* \ln r + C r^2,
\]

where \( A, B, C \) are constants. We write the displacements corresponding to the solution (1)

\[
u = \frac{1}{E} \left[ -\frac{A(1+\nu)}{r} + 2B(1-\nu) \ln r - B(1+\nu)r + 2C \times \right.
\]

\[
\times (1-\nu) r \bigg] + C_1 \sin \theta + C_2 \cos \theta; \quad \nu = \frac{4B^* \theta}{E} + C_1 r + C_2 \cos \theta - C_3 \sin \theta.
\]

Here \( E \) is the material elastic modulus taken equal to one; \( \nu \) is the Poisson ratio; and \( C_1, C_2, C_3 \) are constants determined from the condition of rigid clamping of points on the axis \( \theta = 0 \) that coincides with the lower boundary of the computational domain.

We take radial displacements of its arc with radii \( r_1, r_2 \) at points on the axis of symmetry as nodal displacements of the element and we form the vector of the nodal displacements

\[
\{q\} = \{q_1 q_2\}^T
\]

where \( T \) is the transposition operation. We are given the boundary conditions on the element in the form.


The homogeneity of this latter from condition (4) permits construction of appropriate relationships for the finite element in the mode of the displacement method on the basis of the Lagrange variational equation [2].

Displacements of points of the element in terms of its nodal values are determined, with (1), (2), and (4) taken into account, by the matrix relationship

\[
\{u\} = [H]\{\varphi\} + \begin{bmatrix} C_s \cos \theta \\ -C_s \sin \theta \end{bmatrix}. 
\]

Here \(\{u\} = \{uv\}^T\); \([H]\) is the matrix of mode functions with components \(H_{1j}, H_{2j}, j = 1, 2\), where

\[
\begin{align*}
H_{11} &= \frac{1}{\Delta} \left( a_{1}^2 r + a_{2}^2 r \ln r + a_{3}^2 \frac{1}{r} \right); \quad H_{12} = \frac{1}{\Delta} a_{2}^2 r \theta; \\
a_{11} &= p_0 \left[ p_1 \frac{r_1}{r_2} \ln r_1 + p_2 \frac{r_2}{r_1} \ln r_2 + \frac{p_2}{p_1} \left( \frac{r_2}{r_1} - 1 \right) - 2 \ln r_2 \right]; \\
a_{12} &= p_0 \left[ p_2 \frac{r_1}{r_2} \ln r_2 + p_1 \frac{r_2}{r_1} \ln r_1 + \\
&\quad + \frac{p_2}{p_1} \left( \frac{r_2}{r_1} - 1 \right) - 2 \ln r_1 \right]; \\
a_{22} &= p_0 \left[ p_2 \left( 1 - \frac{r_2}{r_1} \right) - p_1 \left( 1 - \frac{r_1}{r_2} \right) \right]; \\
a_{31} &= 4 p_1 \left[ p_2 r_1 \left( \frac{r_1}{r_2} - 1 \right) + r_1 \left( r_1 - r_2 \right) \right] = 8 \left[ p_2 \left( 1 - \frac{r_2}{r_1} \right) + p_1 \left( 1 - \frac{r_1}{r_2} \right) \right]; \\
a_{32} &= 8 \left[ p_1 \left( 1 - \frac{r_2}{r_1} \right) + p_2 \left( 1 - \frac{r_1}{r_2} \right) \right]; \\
p_0 &= 4 p_2; \quad p_1 = 1 + \nu; \quad p_2 = 1 - \nu; \\
\Delta &= 2 \left[ 2 p_1 \left( \frac{r_1^2 - r_2^2}{r_2} \right) + r_2^2 \left( 1 + 2 \ln r_2 \right) \right] + (1 + \nu^2) (1 + 2 \ln r_1) r_1 + \\
&\quad + 2 p_2 \left[ \ln r_2 \left( 2 - \frac{r_2}{r_1} \right) - 2 \ln r_1 - 4 r_1 (\ln r_2 + \nu \ln r_1) \right].
\end{align*}
\]

The strain vector of the elements is

\[
\{\varepsilon\} = \{e_{rr} e_{\theta \theta}\}^T = [A_e] [H]\{\varphi\} = [B]\{\varphi\},
\]

where

\[
[A_e] = \begin{bmatrix}
\frac{\partial}{\partial r} & 0 \\
\frac{1}{r} & 1 \frac{\partial}{\partial \theta}
\end{bmatrix}
\]