We consider the time-inversion operation for relativistic first-order equations that are constructed without the use of multiple representations of the proper Lorentz group. It is proved that the matrix $T$ of the antilinear time-inversion operation $\psi'(x') = T\bar{\psi}(x)$ is symmetric (antisymmetric) for particles with integral (half-integral) spin. It is shown that for particles with integral (half-integral) spin there exists (does not exist) a linear transformation as a result of which the time-inversion operation reduces to the complex conjugate: $\psi'(x') = \alpha^*\psi^*(x)$ where $\alpha^*$ is a phase factor. A general expression for such a transformation is obtained.

In [1] it was shown that the charge conjugation matrix is symmetric (antisymmetric) for particles with integral (half-integral) spin, and also that we can always choose a basis of the representation in which charge conjugation reduces to complex conjugation. The goal of the present article is an investigation of analogous properties of the time-inversion operation.

Wave functions $\psi, \bar{\psi}$ of a particle in an electromagnetic field with 4-potential $A_\kappa$ satisfy relativistic first-order equations

$$\left(\hat{\partial} - ie\hat{A} + m\right)\psi(x) = 0, \quad (1)$$
$$\bar{\psi}(x)\left(\hat{\partial} - ie\hat{A} - m\right) = 0, \quad (1')$$

where $\hat{\partial} = i\gamma^\kappa \partial_\kappa, A = (A, iA_\theta); \kappa = 1, 2, 3, 4; \partial_\kappa = \partial/\partial t$. The conjugate function $\bar{\psi}$ is defined as $\bar{\psi} = \psi^*\eta$, where $\eta$ is a Hermitian bilinear form matrix that satisfies the following properties [1]:

$$|\eta| = 0, \eta^* = \eta, \gamma^a\gamma^a = -\eta$$

or

$$\gamma^a\eta^a = -\gamma^1\eta^1 (a = 1, 2, 3), \quad (2')$$

From the invariance of (1), (1') with respect to the antilinear time-inversion operation [2]

$$\psi(x) \rightarrow \psi'(x') = T\bar{\psi}(x),$$
$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x') = \psi(x)T^{-1},$$

we obtain the conditions

$$T^{-1}\gamma^aT = -\gamma^a, \quad T^{-1}\gamma^4T = \gamma^4,$$

imposed on the time-inversion matrix $T$ (- denotes conjugation). Functions $\psi'(x')$ and $\bar{\psi}'(x')$ must be related in the same way as functions $\psi(x)$ and $\bar{\psi}(x)$. This leads to the condition

$$T^+\eta T\eta = 1.$$
We consider irreducible equation (I). Then the aggregate of four matrices $\gamma^\nu$ of this equation must be irreducible. By Schur's lemma (see, for example, [3]) this implies that conditions (4) define matrix $T$ up to a constant factor, and condition (5) fixes this factor up to a phase.

Using (3), we introduce the time-inversion operator

$$K_T = T^\dagger K,$$

where $K$ is the complex conjugate operator [4, 5]. Here condition (5) can be interpreted as the condition of the antiunitarity of time-inversion operator $K_T$ with respect to the scalar product $(\psi, \phi) = \psi^* \phi$ by inner [5] variables

$$(K_T\psi, K_T\phi) = (T^\dagger T^\dagger, T^\dagger T^\dagger) = \psi^* T^\dagger T^\dagger = \psi T^\dagger = (\phi, \psi).$$

In [6] it is shown (see also [7]) that if in space $L_\lambda$ of an irreducible representation of the quantum mechanical Poincaré group of spin $s$, not containing reflections, we define an antiunitary time-inversion operation $K_T$, then equation $K_T^2 = (-1)^{2s}$ takes place. It is also shown that an alternative possibility $K_T^2 = -(-1)^{2s}$ is realized in space $L_\lambda^0 \oplus L_\lambda$. We shall prove that in the case when (1) describes a particle with spin $s$, with a wave function given in the space of a finite-dimensional reducible representation of the proper Lorentz group that consists of linking [3] irreducible representations among which there are no multiples, then for operator $K_T$ (6) we have the relation

$$K_T^2 = (-1)^{2s}. $$

We shall consider subspace $L_{\lambda\lambda'}$ of irreducible representation $T_{\lambda\lambda'}^\lambda$ of the proper Lorentz group introduced in the linking scheme. Let $\phi_\lambda(q = -\lambda, -\lambda + 1, \ldots, +\lambda)$ be a standard basis (see, for example, [3, 5, 7]) of an irreducible representation of group $SO(3)$ of weight $\lambda$:

$$T(q, q^*) = \sum_{\lambda} T^\lambda_{\nu\mu}(q) \phi_\lambda(q) \phi^*_\lambda(q^*),$$

where $\phi_\lambda(q)$ denotes the analytic continuation of the matrix of the irreducible representation of group $SO(3)$ of weight $\lambda$ (see (9)) into the domain of complex vector parameters $q$ of the Lorentz group [3].

For an antiunitary time-inversion operator $K_T$ we have the relation ([7], Vol. 2)

$$T(q, q^*) K_T = K_T T(q^*, q).$$

Taking into account (10) and (11) and the antilinearity of $K_T$ (6), we get

$$T(q, q^*) K_T (\phi_\lambda(q) \phi^*_\lambda(q^*)) = \sum_{\nu\mu} T^\lambda_{\nu\mu}(q^*) T^\mu_{\nu\lambda}(q) K_T (\phi_\lambda(q) \phi^*_\lambda(q^*)).$$

But for $T^\lambda_{\nu\mu}(q)$ we get the equality

$$T^\lambda_{\nu\mu}(q^*) = (-1)^{\lambda - \mu} T^\lambda_{-\nu,-\mu}(q),$$

which we can verify by easily transforming the proof introduced in [7] for the case of a real vector parameter $c$. From (10), (12), and (13) it follows that the set of vectors $K_T (\phi_\lambda(q) \phi^*_\lambda(q^*))$ forms under the action of the operators a representation of the proper Lorentz group as a set of vectors $(-1)^{\lambda + \lambda' + \nu'} (\phi_\lambda \phi^*_\lambda)$. By a known lemma ([4], p. 402) sets of vectors $\phi_\lambda \phi^*_\lambda$ and $K_T (\phi_\lambda \phi^*_\lambda)$ are either linearly independent or can be linearly expressed by each other, i.e.,

$$K_T (\phi_\lambda \phi^*_\lambda) = \sum_{\nu\mu} (-1)^{\lambda + \lambda' + \nu'} (\phi_\lambda \phi^*_\lambda) X_{\nu\mu},$$

where

$$(K_T\psi, K_T\phi) = (T^\dagger T^\dagger, T^\dagger T^\dagger) = \psi^* T^\dagger T^\dagger = \psi T^\dagger = (\phi, \psi).$$

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