It is shown that the ring of endomorphisms of an arbitrary free R-module is right self-injective if and only if R is quasi-Frobenius.

It is well known (see [2], [4]), that the ring $R_n$ of matrices over a ring R is left self-injective if and only if R is left self-injective. The matrix ring can be thought of as the ring $\text{End}_R F$ of endomorphisms of a free left R-module F with a finite basis. If we write homomorphisms to the left of elements, then the endomorphism ring $\text{End}_R F$ is right self-injective if and only if $R_n$ is left self-injective. We consider the following question: When is the endomorphism ring $\text{End}_R F$ of a free left R-module F on an infinite set of cardinality $\omega$ right self-injective? The ring $\text{End}_R F$ is anti-isomorphic with the ring $R\omega$ of row-finite matrices of size $\omega$. Thus it suffices to show that the ring $R\omega$ is left self-injective. Our main result (Theorem 4) says that the ring $R\omega$ is left self-injective if and only if the left R-module F is injective. Then with the aid of [3] it follows (Theorem 5) that the ring of endomorphisms of an arbitrary free left R-module is right self-injective if and only if the ring R is quasi-Frobenius.

In the category $R\mathbb{M}$ of left R-modules we define functors, taking values in the category $R\mathbb{M}$ of left modules over the ring $R\omega$ of all row-finite matrices of size $\omega$ ([5], p. 35) with entries from the ring R.

1°. Each R-module A determines a corresponding module $\widehat{A}^\omega$, whose additive group is the direct sum of $\omega$ copies of the Abelian group $(A, +)$ and in which multiplication by an element $\lambda = (\lambda_{\alpha\beta}) \in R$ is given by the formula

$$\lambda a = \left( \sum_\beta \lambda_{\alpha\beta} a_\beta, \sum_\beta \lambda_{\alpha\beta} a_\beta, \ldots, \sum_\beta \lambda_{\alpha\beta} a_\beta, \ldots \right),$$

$$a = (a_1, a_2, \ldots, a_\omega, \ldots) \in A^\omega.$$  (1)

Each R-homomorphism $\varphi : A \rightarrow B$ determines a homomorphism $\overline{\varphi}^\omega$ by the formula

$$\overline{\varphi}^\omega (a) = (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_\omega), \ldots).$$  (2)

It is easy to check that $\overline{\varphi}$ is an $R\omega$-homomorphism. Each of these functors can be defined from the category $R\mathbb{R}$ of right R-modules into the category of right $R\omega$-modules $R\mathbb{R}$.

2°. Each R-module A determines a corresponding module $\widehat{A}^\omega$, whose additive group is isomorphic to the direct product of $\omega$ copies of the group $(A, +)$, with multiplication by elements from $R\omega$ given by Eq. (1) and for homomorphisms $\varphi : A \rightarrow B$ of left R-modules, the mappings $\varphi^\omega(a)$ are defined by the equation

$$\varphi^\omega (a) = (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_\omega), \ldots)$$  (2')

Let $\psi : R \rightarrow R\omega$ assign to an arbitrary element r of the ring R the matrix $\psi(r) \in R\omega$, having all diagonal entries equal to r and all other entries zero. If A is an arbitrary left $R\omega$-module, it may be considered as an R-module, setting $ra = \psi(r)a$ for arbitrary $a \in A$. 


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LEMMA 1. If the left R-module A is projective, the left R_w-module A^w is also projective. If the right R-module B is projective, the right R_w-module B^w is also projective.

Proof. The left R_w-module R^w is a direct summand of the free R_w-module R_w. A projective R-module A is a direct summand of a free module, hence \( \Sigma A \cong A \oplus A' \) for some left R-module A'. Therefore \( \Sigma A(R^w) \cong \Lambda^w \oplus \Lambda^w \) and \( \Lambda^w \) are projective R-modules. Similarly, the right R_w-module R_w is a direct summand of the free right R_w-module R_w. From the decomposition \( \Sigma B(R^w) \cong B_w \oplus B' \), it follows that \( \Sigma B(R^w) \cong B^w \oplus B'^w \), so that the module B^w is a direct summand of the projective right R_w-module \( \Sigma B(R^w) \). Thus B^w is a projective right R_w-module.

The free left R-module F with basis x_1, x_2, ..., x_w, ... may be considered as a right R_w-module. For any \( \lambda = (\lambda_{\alpha\beta}) \in R_w \) there exists an endomorphism L of the module F such that \( L(x_{\alpha}) = \Sigma_{\beta} \lambda_{\alpha\beta} x_{\beta} \). For any element \( a \in F \) set

\[ a\lambda = L(a) \]

LEMMA 2. If F is a free left R-module and if R_w is the ring of all row-finite matrices of size w over R, then the right R_w-modules F and R^w are isomorphic.

Proof. Let x_1, ..., x_w, ... be a free basis of the R-module F. Each element \( a \in F \) determines a row \( \varphi(a) = (\mu_1, \mu_2, ..., \mu_w, ...) \in \tilde{R}^w \) where \( a = \Sigma \mu \lambda x^w \) and therefore only finitely many elements \( \mu \lambda \) are non-zero. The mapping \( \varphi \) is clearly an isomorphism of the Abelian groups \( F \) and \( \tilde{R}^w \). Let \( (\lambda_{\alpha\beta}) \) be an arbitrary element of the ring \( R_w \). There exists an endomorphism L of the module F such that

\[ L(a) = L(A) = \Sigma \mu \lambda x^w \]

We have

\[ \varphi(L(a)) = \Sigma \mu \lambda x^w \]

and the lemma is proved.

COROLLARY. A free left R-module F is a projective right R_w-module.

PROPOSITION 3. For arbitrary left R-modules A and B there is a natural isomorphism

\[ \text{Hom}_R(\tilde{A}^w, B) \cong \text{Hom}_R(A, B) \]

Proof. Let \( \tilde{f} \in \text{Hom}_{R_w}(\tilde{A}^w, B^w) \) and let \( a 
\in A^w \) be an arbitrary element. Then \( \tilde{f}(a) = \Sigma \lambda_{\alpha\beta} \tilde{f}(a) \), where the summation sign has meaning since at most finitely many elements \( \alpha \lambda \) are different from zero. Noting that the summation is produced by only those indices \( \alpha \) for which \( a\alpha \neq 0 \), we have

\[ \tilde{f}(a) = \Sigma \lambda_{\alpha\beta} \tilde{f}(\alpha) = (\Sigma \lambda_{\alpha\beta} \tilde{f}(\alpha)) \]

where we denote by E the matrix in the ring R_w, having the elements \( e_{\alpha\beta} \) in the rows numbered \( \alpha \) for which \( a\alpha \neq 0 \).

Let \( p_1 : A \rightarrow \tilde{A}^w \) and \( p_2 : B \rightarrow B^w \) denote the injection maps of the modules A and B into the first coordinates of the R-modules \( \tilde{A}^w \) and \( B^w \), respectively. Further, let \( \pi_1 : \tilde{A}^w \rightarrow A \) and \( \pi_2 : B^w \rightarrow B \) be the projections of the modules \( \tilde{A}^w \) and \( B^w \) defined by the equations

\[ \pi_1((a_1, a_2, ..., a_w, ...)) = a_1, \quad \pi_2 ((b_1, b_2, ..., b_w, ...)) = b_1 \]

For an element \( \tilde{f} \in \text{Hom}_{R_w}(\tilde{A}^w, B^w) \) we define a mapping \( f \in \text{Hom}_R(A, B) \) by the formula \( f(a) = \pi_2 \tilde{f}(p_1(a)) \) for any \( a \in A \). A homomorphism \( f \rightarrow \tilde{f} : \text{Hom}_R(A, B) \rightarrow \text{Hom}(\tilde{A}^w, B^w) \) yields the equality

\[ \tilde{f}(a) = \Sigma \lambda_{\alpha\beta} (p_2 f p_1)(e_{\alpha\beta}) \]