SELF-INJECTIVE RINGS AND ENDOmorphisms  
OF FREE MODULES

V. I. Gemintern  
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It is shown that the ring of endomorphisms of an arbitrary free R-module is right self-injective if and only if R is quasi-Frobenius.

It is well known (see [2], [4]), that the ring $R_n$ of matrices over a ring R is left self-injective if and only if R is left self-injective. The matrix ring can be thought of as the ring $\text{End}_RF$ of endomorphisms of a free left R-module F with a finite basis. If we write homomorphisms to the left of elements, then the endomorphism ring $\text{End}_RF$ is right self-injective if and only if $R_n$ is left self-injective. We consider the following question: When is the endomorphism ring $\text{End}_RF$ of a free left R-module F on an infinite set of cardinality $\omega$ right self-injective? The ring $\text{End}_RF$ is anti-isomorphic with the ring $R_\omega$ of row-finite matrices of size $\omega$. Thus it suffices to show that the ring $R_\omega$ is left self-injective. Our main result (Theorem 4) says that the ring $R_\omega$ is left self-injective if and only if the left R-module F is injective. Then with the aid of [3] it follows (Theorem 5) that the ring of endomorphisms of an arbitrary free left R-module is right self-injective if and only if the ring R is quasi-Frobenius.

In the category $R_\omega$ of left R-modules we define functors, taking values in the category $R_\omega$ of left modules over the ring $R_\omega$ of all row-finite matrices of size $\omega$ ([5], p. 35) with entries from the ring R.

1°. Each R-module A determines a corresponding module $\tilde{A}_\omega$, whose additive group is the direct sum of $\omega$ copies of the Abelian group $(A, +)$ and in which multiplication by an element $\lambda = (\lambda_{\alpha\beta}) \in R$ is given by the formula

$$\lambda a = (\sum_\alpha \lambda_{\alpha 1} a_1, \sum_\beta \lambda_{\alpha 2} a_2, \ldots, \sum_\gamma \lambda_{\alpha\omega} a_\omega, \ldots),$$

(1)

Each R-homomorphism $\varphi : A \rightarrow B$ determines a homomorphism $\tilde{\varphi}_\omega$ by the formula

$$\tilde{\varphi}_\omega(a) = (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_\omega), \ldots).$$

(2)

It is easy to check that $\tilde{\varphi}$ is an $R_\omega$-homomorphism. Each of these functors can be defined from the category $\mathbb{M}_R$ of right R-modules into the category of right $R_\omega$-modules $\mathbb{M}_{R_\omega}$.

2°. Each R-module A determines a corresponding module $\tilde{A}_\omega$, whose additive group is isomorphic to the direct product of $\omega$ copies of the group $(A, +)$, with multiplication by elements from $R_\omega$ given by Eq. (1) and for homomorphisms $\varphi : A \rightarrow B$ of left R-modules, the mappings $\varphi^\omega(a)$ are defined by the equation

$$\varphi^\omega(a) = (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_\omega), \ldots)$$

(2')

Let $\psi : R \rightarrow R_\omega$ assign to an arbitrary element $r$ of the ring R the matrix $\psi(r) \in R_\omega$, having all diagonal entries equal to r and all other entries zero. If A is an arbitrary left $R_\omega$-module, it may be considered as an R-module, setting $ra = \psi(r)a$ for arbitrary $a \in A$.
LEMMA 1. If the left $R$-module $A$ is projective, the left $R^\omega$-module $A^\omega$ is also projective. If the right $R$-module $B$ is projective, the right $R^\omega$-module $B^\omega$ is also projective.

Proof. The left $R^\omega$-module $R^\omega$ is a direct summand of the free $R^\omega$-module $R^\omega$. A projective $R$-module $A$ is a direct summand of a free module, hence $\Sigma^\alpha R \cong A \oplus A'$ for some left $R$-module $A'$. Therefore $\Sigma^\alpha (R^\omega) \cong A^\omega \oplus A'^\omega$ and $A^\omega$ are projective $R^\omega$-modules. Similarly, the right $R^\omega$-module $R^\omega$ is a direct summand of the free right $R^\omega$-module $R^\omega$. From the decomposition $\Sigma^\alpha R^\omega \cong B \oplus B'$ of right $R$-modules, it follows that $\Sigma^\alpha (R^\omega) \cong \tilde{B}^\omega \oplus \tilde{B'}^\omega$, so that the module $\tilde{B}^\omega$ is a direct summand of the projective right $R^\omega$-module $\Sigma^\alpha (R^\omega)$. Thus $B^\omega$ is a projective right $R^\omega$-module.

The free left $R$-module $F$ with basis $x_1, x_2, \ldots, x_\omega, \ldots$ may be considered as a right $R^\omega$-module. For any $\lambda = (\lambda_{\alpha \beta}) \in R^\omega$, there exists an endomorphism $L$ of the module $F$ such that $L(x_\alpha) = \sum_\beta \lambda_{\alpha \beta} x_\beta$. For any element $a \in F$ set

$$a\lambda = L(a).$$

LEMMA 2. If $F$ is a free left $R$-module and if $R^\omega$ is the ring of all row-finite matrices of size $\omega$ over $R$, then the right $R^\omega$-modules $F$ and $\tilde{F}$ are isomorphic.

Proof. Let $x_1, x_2, \ldots, x_\omega, \ldots$ be a free basis of the $R$-module $F$. Each element $a \in F$ determines a row $\varphi(a) = (\mu_1, \mu_2, \ldots, \mu_\omega, \ldots) \in \tilde{F}$ where $a = \Sigma_\omega \mu_\omega x_\omega$ and therefore only finitely many elements $\mu_\omega$ are non-zero. The mapping $\varphi$ is clearly an isomorphism of the Abelian groups $(F, +)$ and $(\tilde{F}, +)$. Let $(\lambda_{\alpha \beta})$ be an arbitrary element of the ring $R^\omega$. There exists an endomorphism $L$ of the module $F$ such that

$$L(a) = L\left(\sum_\alpha \mu_\alpha x_\alpha\right) = \sum_\alpha \mu_\alpha L(x_\alpha) = \sum_\alpha \mu_\alpha \left(\sum_\beta \lambda_{\alpha \beta} x_\beta\right) = \sum_\beta \left(\sum_\alpha \mu_\alpha \lambda_{\alpha \beta}\right) x_\beta.$$ 

We have

$$\varphi(L(a)) = (\sum_\alpha \mu_\alpha \lambda_{\alpha 1}, \sum_\alpha \mu_\alpha \lambda_{\alpha 2}, \ldots, \sum_\alpha \mu_\alpha \lambda_{\alpha \omega}, \ldots)$$

and the lemma is proved.

COROLLARY. A free left $R$-module $F$ is a projective right $R^\omega$-module.

PROPOSITION 3. For arbitrary left $R$-modules $A$ and $B$ there is a natural isomorphism

$$\text{Hom}_{R^\omega}(\tilde{A}^\omega, B^\omega) \cong \text{Hom}_R(A, B).$$

Proof. Let $\tilde{f} \in \text{Hom}_{R^\omega}(\tilde{A}^\omega, B^\omega)$ and let $a \in A^\omega$ be an arbitrary element. Then $\tilde{f}(a) = \Sigma_\alpha \tilde{f}(0, \ldots, 0, a_\alpha, 0, \ldots)$, where the summation sign has meaning since at most finitely many elements $a_\alpha$ are different from zero. Noting that the summation is produced by only those indices $\alpha$ for which $a_\alpha \neq 0$, we have

$$\tilde{f}(a) = \sum_\alpha \tilde{f}(0, \ldots, a_\alpha, \ldots) = \left(\sum_\alpha e_{\alpha 1} a_\alpha\right) \tilde{f}(a) = E a \tilde{f}(a),$$

where we denote by $E a$ the matrix in the ring $R^\omega$, having the elements $e_{\alpha \beta}$ in the rows numbered $\alpha$ for which $a_\alpha \neq 0$.

Let $p_1 : A \rightarrow \tilde{A}^\omega$ and $p_2 : B \rightarrow B^\omega$ denote the injection maps of the modules $A$ and $B$ into the first coordinates of the $R$-modules $A^\omega$ and $B^\omega$, respectively. Further, let $\pi_1 : \tilde{A}^\omega \rightarrow A$ and $\pi_2 : B^\omega \rightarrow B$ be the projections of the modules $\tilde{A}^\omega$ and $B^\omega$ defined by the equations

$$x_1((a_1, a_2, \ldots, a_\omega, \ldots)) = a_1, \quad x_2((b_1, b_2, \ldots, b_\omega, \ldots)) = b_1.$$ 

For an element $\tilde{f} \in \text{Hom}_{R^\omega}(\tilde{A}^\omega, B^\omega)$ we define a mapping $f \in \text{Hom}_R(A, B)$ by the formula $f(a) = \pi_2 \tilde{f}(p_1(a))$ for any $a \in A$. A homomorphism $f \rightarrow \tilde{f} : \text{Hom}_{R}(A, B) \rightarrow \text{Hom}(\tilde{A}^\omega, B^\omega)$ yields the equality

$$\tilde{f}(a) = \sum_\alpha a_\alpha (p_2 \pi_1)(e_{12})$$

for any $a \in \tilde{A}^\omega$. 

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