The concept of a finitary ring is introduced, permitting statement of the conditions under which the concepts of the $\alpha$- and $\beta$-ring coincide.

In [2] V. P. Elizarov introduced the concept of $\alpha$- and $\beta$-rings (by "ring" is always understood "associative ring with an identity element"):

a) Ring $R$ is called an $\alpha$-ring if every element $a \in R$, such that $Ra \neq R \neq aR$, belongs to a proper two-sided ideal of ring $R$;

$\beta$) Ring $R$ is called a $\beta$-ring if each of its maximal (by "maximal" is always understood "proper maximal") two-sided ideals $I$ is completely prime, i.e., from $a_1 \notin I$ and $a_2 \notin I$ it follows that $a_1a_2 \notin I$.

In this article we shall show that for some rings (for example, finite local rings) the concepts of $\alpha$- and $\beta$-rings coincide (compare with the assertion for a ring of triangular $2 \times 2$ matrices on p. 229 of [2] which is anti-isomorphic to the semigroup algebra considered on p. 227 of [2]).

For a left ideal $L$ of ring $R$ with $a \in R$, the symbol $L : a$ denotes a right fractional ideal of $L$, i.e.,

$L : a = \{ r \in R \mid ra \in L \}$. At that, $L : a = L \subseteq L : a$ for any $a \in R$ if and only if $L$ is a two-sided ideal in $R$, and $L : a \subseteq L$ for any $a \in R \setminus L$ if and only if $L$ is a completely prime ideal [1].

**Lemma 1.** If $W$ is a maximal left ideal of ring $R$, then for any $a \in R \setminus W$, $W : a$ is a maximal left ideal in $R$, and $W = (W : a) : b = W : ba$ for a suitable $b \in R$.

**Proof.** Let there exist a left ideal $L$ such that $W : a \subseteq L \subseteq R$. Then

$$1 = ba + w$$

for suitable elements $b \in L$ and $w \in W$. Since

$L \setminus (W : a) \neq \emptyset$, to $La \subseteq W$ and $(La, W) = R$.

Consequently,

$$(1 - ab)a = aw \in W \text{ and } 1 - ab \in W : a \subseteq L.$$ 

Therefore $1 \in L$ and $L = R$, contrary to the condition given. From $(\ast)$ we obtain $xba = x - xw \in W$ for any $x \in W$. Thus,

$W \subseteq W : ba \neq R \text{ and } W = W : ba = (W : a) : b$.

The lemma is proven.

Let $W$ be a maximal left ideal of ring $R$. Say $Q_W = \{ W : a \mid a \in R \}$ and $W^0 = \bigcap_{a \in R} (W : a) : b$. We note that $W^0$ is a two-sided ideal.
**Lemma 2.** Let $K_i$ be maximal left ideals, $1 \leq i \leq n$, and let $L$ be a left ideal of ring $R$ such that $K = \bigcap_{i=1}^{n} K_i \subseteq L \neq R$. Then for a suitable index $i_0$ and $\rho \in R$ the following relations are satisfied:

$$K : \rho = K_{i_0} : \rho = L : \rho = R.$$  

**Proof.** Suppose that $\bigcap_{i=1}^{n} K_i$ is irreducible, i.e., $\bigcap_{i=1}^{n} K_i \setminus K_j \neq \emptyset$ for any $j$, $1 \leq j \leq n$. If $K_{i_0} \subseteq L$ for some $i_0$, then $K_{i_0} = L$, and (***) apparently is satisfied for any $\rho \in \bigcap_{i \neq i_0} K_i \setminus K_{i_0}$. If $K_i \not\subseteq L$, then for $\rho_{n-1} \in K_n \setminus L$ we have $K : \rho_{n-1} = \bigcap_{i=1}^{n-1} K_i : \rho_{n-1} \subseteq L : \rho_{n} \neq R$ and we may carry out the proof by induction. The lemma is proved.

**Lemma 3.** Let $W$ be a maximal left ideal of ring $R$ such that $Q_w$ is finite. Then for any $c \in R \setminus W^0$ there exists an element $d \in R$ such that $W^0 : dc = W$.

**Proof.** Since $W^0 : c = \bigcap_{a \in R} (W : ca)$, then, having substituted $K = W^0 : c, K_i = W : ca_i, \rho = d'$, in Lemma 2, we see that there exists an element $d' \in R$ such that $W^0 : d'c = W : d'ca$.

According to Lemma 1 there exists a $b \in R$ such that $W : dc = W^0 : bd'c = W : bd'ca = W$, where $d = bd'$. The lemma is proved.

**Assertion 1.** Let $W$ be a maximal left ideal of ring $R$ such that $Q_w$ is finite. Then $W^0$ is a maximal two-sided ideal in $R$.

**Proof.** Suppose that there exists a two-sided ideal $I$ such that $R \neq I \supseteq W^0 = \bigcap_{a \in R} (W : a)$. Then for a suitable $a \in R$ there exists an element $c \in R$ such that $R \neq I : c \supseteq W : ca$.

(it is sufficient to substitute $L = I, K = W^0, K_j = W : a_j, \rho = c$ in Lemma 2). Because of maximality $I : c = W : ca$ and $I : d = W$ for a suitable $d \in R$. But $I \subseteq I : d = W$, i.e., $I \subseteq W^0$ and $I = W^0$. The assertion is proved.

We shall say that ring $R$ is (left) finitary if for any maximal (left) ideal $W$ the set $Q_w$ is finite. Clearly, every commutative ring is finitary. If all prime left $R$-modules over ring $R$ are finite, then ring $R$ is left finitary. (In particular, any finite ring is finitary.)

A simple consequence of Assertion 1 is:

**Assertion 2.** $I$ is a maximal two-sided ideal of finitary ring $R$ if and only if $I = W^0$ for some maximal left ideal $W$ of ring $R$.

Since, moreover, $I : d = W$ for a suitable $d \in R$, we immediately obtain:

**Theorem.** A finitary ring $R$ is a $\beta$-ring if and only if every maximal left ideal of ring $R$ is two-sided.

In conclusion, we introduce two corollaries:

**Corollary 1.** A finitary $\beta$-ring is an $\alpha$-ring.

Actually, any left ideal $Ra = R$ can be extended to a maximal left ideal $W \supseteq Ra$ which is two-sided.

We note that in this case the conceptions of right and left nonidentity elements coincide [2].