IMPROVEMENT OF REMAINDER TERM FOR THE DIVISORS PROBLEM

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New bounds are obtained for the remainder term in the divisors problem.

A series of authors have occupied themselves with the problem of determining the remainder for the divisors problem (see [3], p. 157). The last result is that of Ch'ih-Tsiang-T'U and Richert, who showed that \( \Delta(R) \ll R^{12/37} \), where \( \Delta(R) \) is the remainder term for the divisors problem. In the present work it is shown that \( \Delta(R) \ll R^{12/37} \ln^{62/37} R \). The improvement is gained by computing more refined bounds with double trigonometric sums. As distinct from [1], [2], and other works, the trigonometric sums are calculated without converting them to an integral and estimating that.

**Lemma 1.** Let \( f(z) \) be a function, analytic for

\[ |z - x| \leq \sqrt{M_1 \ln X}, \quad X < x \leq X_1 < 2X; \]

let \( f(x) \) be a real function,

\[ 0 < M_1^{-1} \leq f_x(x) \leq M_1^{-1}; \]

\[ f_{k+2}^{(k, x)}(x) \ll k! (MX^k)^{-1}, \quad k = 1, 2, \ldots; \]

\[ X^3 \gg \ln^3 X \cdot M_1^3 M^{-1}; \quad M_1 \gg M \gg M_0. \]

Then

\[ \sum_{X < x < X_1} e^{2\pi i f_x(x)} = e^{\phi_1} \sum_{f(x) \leq \gamma_0} (f'(x_n))^{-1/2} \times e^{2\pi i f(x_n - f_n)} + O\left(\sqrt{M_1 + \ln^3 X}\right), \]

where \( f_{X_1}^1(X_0) = n \).

We prove a lemma similar to Lemma 2, p. 121 of [4].

**Lemma 2.** Let

\[ 0 < M_1^{-1} \leq f_x(x, y) \leq M^{-1}; \]

\[ f_{k+2}^{(k, x)}(x, y) \ll k! (MX^k)^{-1}, \quad k = 1, 2, \ldots; \]

\[ X^3 \gg \ln^3 X \cdot M_1^3 M^{-2}; \quad M \gg M_0; \]

\( f_{X^2}(x, y) \) be piecewise monotonic on a finite number of connected regions such that along with the two points \( x(x_1, x_2) \) and \( y(y_1, y_2) \) it also contains the points \( \alpha x + (1 - \alpha) y \), if \( x_1 = y_1 \) or \( x_2 = y_2 \), where

\[ 0 \leq \alpha \leq 4; \quad f_{xy}(x, y) \ll X(YM)^{-1}; \]

\[ M^{-1} \leq f_x(x, y) f_{xy}(x, y) - (f_{xy}(x, y))^2 \ll X^3 (YM)^2, \quad M_1 \gg M_0. \]


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Then
\[ \sum_{D} e^{2 \pi i (x, y)} \ll \ln N \left( \frac{x^2 + x}{M} + X \sqrt{M} M^{-1} + Y \sqrt{M} + Y \ln^2 X \right), \]
where the summation is carried out on the domain D:
\[ X \leq x \leq X_1 \leq 2X, \quad Y \leq y \leq Y_1 \leq 2Y, \quad xy \leq N, \]
and
\[ \sum_{D'} e^{2 \pi i (x, y)} \ll \ln N \left( e^{Y^2} M^{-1} + Y \sqrt{M} + Y \ln^2 X \right), \]
where \( D' \) is the subdomain of D in which \( |x/y - \gamma_1| \leq \varepsilon \).

**Proof.**
\[ \sum_{D_i} e^{2 \pi i (x, y)} \ll \ln X \left| \sum_{D_i} e^{2 \pi i (x, y)} \right|, \]
where \( D_i \) is the subregion of one of the regions indicated in the conditions of the lemma, in which
\[ (M')^{-1} \leq f_n(x, y) \leq 2(M')^{-1}; \quad M \leq M' \leq M_1. \]
Applying Lemma 1, we get
\[ \sum_{D_i} e^{2 \pi i (x, y)} = \sum_{i=1}^{N} e^{2 \pi i/\lambda} \sum_{h \in M} (f_n(x, y))^{2 \pi i/\lambda} e^{2 \pi i ((x, y) - \gamma h)} \]
\[ + O(Y \sqrt{M} + Y \ln^2 N) \ll \frac{x^2}{M} + \frac{Y \sqrt{M}}{M} + Y \sqrt{M_1} + Y \ln^2 N. \]

The second part of the lemma is obvious.

We prove some assertions regarding the roots of polynomials of the form
\[ f(\theta) = \sum_{i=0}^{m} a_i \theta^i \sum_{\sum_{i=1}^{n} + \sum_{i=2}^{n} + \sum_{i=3}^{n} = m} a_i, b_i, h_i, g_i, h_i, g_i. \]
where
\[ \sigma_1 = \sigma_1 + \sigma_2 + \sigma_3; \quad \sigma_2 = \sigma_2 + \sigma_2 + \sigma_3; \]
\[ \sigma_3 = \sigma_3 + \sigma_3 + \sigma_3; \]
\[ \alpha_1 = h_1/h_2; \quad \alpha_2 = h_3/h_4; \quad \alpha_3 = h_5/h_6; \]
\[ R_f(\theta), \phi(\theta) \text{ is the resultant of the polynomials } f(\theta) \text{ and } \phi(\theta); \]
\[ H_2 = h_2 h_4 h_6; \quad Q = q_5 q_6 q_8; \quad Q_2 = q_5 q_6 q_8. \]

**LEMMA 3.** Let
\[ f(0) = \theta^3 + a_1 \theta^2 + a_2 \theta + a_3, \]
\[ \phi(0) = \theta^3 + b_1 \theta^2 + b_2 \theta + b_3. \]
Then, if
\[ R_{R_f(x, y), R_{\phi(x, y)}} \neq 0 \text{ and } \lim_{x \to \infty} \max_{a_1, a_2, a_3} \frac{R_n(\tau_1, \tau_2, \tau_3)}{R_{\phi(a_1, a_2, a_3)}} \neq 0, \]
then
\[ \frac{1}{Q} \sum_{h} H_2^2 \ll \frac{1}{Q} \sum_{h} H_2^2 + Y \sqrt{Y^2 X^2}, \]
where the summation is conducted according to \( 1 \leq |h_1| \leq \alpha_1 = 1 \) such that
\[ f(0) = \phi(0 + \varepsilon); \quad f(\theta) \neq \phi(\theta + \varepsilon); \quad \varepsilon \ll \theta_1; \]
\[ R_1(\alpha_1, \alpha_2, \alpha_3) = R_{\phi(\alpha_1, \alpha_2, \alpha_3)}(\alpha_1, \alpha_2, \alpha_3); \quad \alpha < 1; \]
\[ (\text{if } \phi(0) \equiv \phi(0) \text{, then } R_1(\alpha_1, \alpha_2, \alpha_3) = R_{\phi(\alpha_1, \alpha_2, \alpha_3)}; \quad \alpha < 1; \]
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