The present survey is mainly devoted to works published from 1969-1980 in Ref. Mat. Zh. in which completely positive linear mappings are studied that arise, in particular, in the quantum theory of open systems, the quantum theory of measurements, and in problems of the dynamics of a small system interacting with a large system. Here the probabilistic aspect is singled out, and analogies and connections with ordinary Markov processes are indicated.

In this survey we consider a number of mathematical works of recent years in which completely positive linear mappings of algebras with an involution are studied.

Interest in this topic was stimulated not only by the development of the theory of C*-algebras but also by some mathematical problems of quantum physics. Such mappings arise in the quantum theory of open systems and the quantum theory of measurements. We note specially the problem on the dynamics of a small system interacting with a large system. The study of semigroups of completely positive linear mappings is connected with the rigorous derivation of the quantum kinetic equation in a series of works of Davies [59, 60, 61, 62, 64, 69, 75].

He introduced the concept of a quantum stochastic process which generalizes the concept of a Markov process. Lindblad [137, 138] defined and studied non-Markov quantum stochastic processes.

Accardi [1, 33, 34, 35] expressed another point of view regarding a "noncommutative quantum stochastic process." He attempted to overcome difficulties indicated already in the work of Pilis (1966) by a new definition of the concept of conditional mathematical expectation in the noncommutative case. We shall clarify the difficulties concerned. Let \( \omega_1, \omega_2, \ldots, \omega_k \) be a homogeneous Markov chain with state space \( X \). We denote by \( a_1 \otimes \cdots \otimes a_k \) the tensor product \( a_1(\omega_1) \cdots a_k(\omega_k) \) of functions \( a_1, \ldots, a_k \) on the space \( X \). We define the functional \( p_k(a_1 \otimes \cdots \otimes a_k) = M[a_1(\omega_1) \cdots a_k(\omega_k)] \). If \( T(a) = M[a(\omega)/\omega_1 = x] \), then \( p_k(a_1 \otimes \cdots \otimes a_k) = p_1(a_1 T(\cdots T(a_k) \cdots )) \). It would seem that this formula could be carried over directly to the noncommutative case: it suffices to replace the algebra of complex functions on \( X \) by an algebra with involution, the operator \( T \) of the Markov chain by a completely positive linear mapping \( T \) of this algebra into itself, and the functional \( p_1 \) by a positive, linear, normalized functional \( p_1 \). However, \( p_k(a_1 \otimes \cdots \otimes a_k) \) may be negative for positive elements \( a_1, \ldots, a_k \).

We note further that the operator \( T \) of the Markov chain is a positive, linear mapping. Now if the direct product of Markov chains is a Markov chain and the tensor product of the operators of the chains is a positive mapping, then for noncommutative algebras with an involution the tensor product of two positive linear mappings need not be positive. The introduction of the concept of a completely positive mapping is closely connected with this circumstance, since on passing to completely positive mappings "stability relative to the operation of tensor product" is restored. In the commutative case these concepts coincide.
Here we have tried to distinguish the probabilistic aspect and to indicate analogies and connections with ordinary Markov processes, in particular, with random walks on noncommutative groups.

The bibliography of the survey contains mainly works published from 1969 to 1980. A number of books on quantum theory are also indicated there.

1. Completely Positive Linear Mappings

1.1. Definition. Let \( \mathfrak{A} \) be a \( \mathbb{C}^* \)-algebra, and let \( M_n(\mathfrak{A}) \) be the algebra of matrices of order \( n \times n \) with elements in \( \mathfrak{A} \). A linear mapping \( T: \mathfrak{A} \rightarrow \mathfrak{A} \) can be extended to a linear mapping \( T_n:M_n(\mathfrak{A}) \rightarrow M_n(\mathfrak{A}) \), by setting

\[
T_n(a_{ij}) = (T(a_{ij})).
\]

If \( T_n(a^*a) \geq 0 \) for any \( a \in M_n(\mathfrak{A}) \), then \( T \) is called \( n \)-positive. A \( 1 \)-positive mapping is a positive mapping. \( T \) is called completely positive if for any \( n \) the mapping \( T_n \) is \( n \)-positive.

1.2. Algebra of Matrices. Form of a Completely Positive Linear Mapping. Let \( \mathfrak{A} \) be an \( M_n \)-algebra of complex matrices of order \( n \times n \) with the usual involution — the Hermitian conjugate, and let \( \{e_i\} \) be the standard orthonormal basis in the linear space \( \mathfrak{A} \) with scalar product \( (a, b) = \sum (b^*a) \).

A mapping \( T: \mathfrak{A} \rightarrow \mathfrak{A} \) is positive if and only if

\[
\sum_{i,j=1}^{n} \sum_{k,l=1}^{n} (Te_{ij}, e_{kl}) a_{ik}a_{lj} \geq 0
\]

for any \( (a_{ij}) \), \( i, j = 1, \ldots, n \).

Hence, by the theorem on the spectral decomposition of Hermitian matrices, \( (Te_{ij}, e_{kl}) \) is a convex linear combination of no more than \( n^2 \) functions of the form \( \overline{e_{kl}}^*e_{ij} \); therefore,

\[
T(a) = \sum_{i=1}^{n^2} \alpha_i^*a\alpha_i.
\]

Thus, this formula gives the general form of a completely positive linear mapping \( T:M_n \rightarrow M_n \).

We note that it may be assumed that \( (\alpha_i, \alpha_j) = 0 \) for \( i \neq j \).

As an example we consider the linear mapping

\[
T(a) = \sum_{i,j=1}^{n} p_{ij}e_{ij}ae_{ij},
\]

where \( e_{ij}e_{jk} = e_{ij} \), \( e_{ij}e_{ik} = 0 \) for \( k \neq j \) and \( \{e_i\} \) is an orthonormal basis in \( \mathbb{C}^n \). Condition (1) is then equivalent to the condition

\[
\sum_{i,j=1}^{n} p_{ij}(ae_{ij}, e_{ij})^2 \geq 0,
\]

i.e., the mapping \( T \) defined by formula (2) is completely positive if and only if \( p_{ij} \geq 0 \) for all \( i \) and \( j \). Since

\[
T(a) = \sum_{i,j=1}^{n} p_{ij}(ae_{ij}, e_{ij})e_{ii},
\]

the image of the algebra \( \mathfrak{A} \) under the mapping \( T \) is contained in the commutative subalgebra generated by the projectors \( e_{ii} \), \( i = 1, 2, \ldots, n \). Therefore, the study of \( T_k \) reduces to the study of powers of the positive matrix \( (p_{ij}) \). However, if, for example, we consider a mapping \( T_1 \) of the form

\[
T_1:a \rightarrow T(u^{-1}au),
\]

where \( u^{-1} = u^* \), and \( T \) is given by formula (2), then the question of the properties of \( T_k^* \) as \( k \rightarrow \infty \) becomes nontrivial.

1.3. Bistochastic Mappings of an Algebra of Matrices. A completely positive linear mapping \( T \) we call stochastic if \( T(1) = 1 \), where \( 1 \) is the identity of the \( \mathbb{C}^* \)-algebra. A stochastic mapping \( T \) of an algebra of matrices we call bistochastic if \( \text{tr}(T(a)) = \text{tr}(a) \).