TARGET SPACE DUALITY OF CALABI–YAU SPACES WITH TWO MODULI

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In the context of superstring compactifications on Calabi–Yau threefolds, we consider the Picard–Fuchs equations that are obeyed by the periods of the holomorphic three-form. We review, focusing on an example with two moduli, some powerful algebro-geometric techniques for computing the monodromy group of these equations, which is closely related to the target space duality group. For the example investigated, the latter is shown to be given by a three-dimensional representation of a central extension of $B_5$, the braid group on five strands.

1. INTRODUCTION AND SUMMARY

In this contribution, based on [1], we aim at describing some useful algebraic geometric tools for computing the target space duality group acting on the moduli space $M$ of string theory. This is the space of deformations by truly marginal operators of the underlying $c = 9$ $(2, 2)$ supersymmetric conformal field theory, and is known to classify inequivalent superstring vacua. The knowledge of its local and global properties is essential towards the full comprehension of the symmetries and the physical couplings of the low-energy effective lagrangian obtained by compactification of the string down to four dimensions. It has been known for some time that when the six extra dimensions are interpreted geometrically as forming a Kähler manifold of vanishing first Chern class (Calabi–Yau manifold) [2], the moduli parameters reflect the freedom of deforming both its complex and Kähler structure, and their number is given by the dimensions $b_{21}$ and $b_{11}$ of respectively the $H^{(2,1)}$ and $H^{(1,1)}$ complex cohomology groups [3]. Thus, as can be inferred from both supersymmetry and effective field theory arguments [4, 5], the moduli space splits into two factors $M = M_{(2,1)} \otimes M_{(1,1)}$, which are exchanged by “mirror symmetry” [6–8]. The local geometry of each factor is “special” in the sense that, due to $N = 2$ spacetime supersymmetry [10], their curvature tensor satisfies the constraint [11–14]

$$R_{\alpha \beta \gamma \delta} = g_{\alpha \beta} g_{\gamma \delta} + g_{\alpha \gamma} g_{\beta \delta} - \varepsilon^{\gamma \delta \rho \sigma} g_{\alpha \beta} W_{\alpha \gamma \rho} W_{\beta \delta \sigma} g^{\rho \sigma}. \quad (1)$$

Here, $g_{\alpha \beta}(z, \bar{z}) = \partial_{\alpha} \partial_{\beta} K(z, \bar{z})$ is the Kähler metric on either factor of the moduli space, and the (completely symmetric) tensors $W_{\alpha \gamma \rho}$ are holomorphic functions of the moduli $z^{\alpha}$ ($\alpha = 1, \ldots, n$, where $n$ is either $b_{11}$ or $b_{21}$, although we will consider $n = b_{21}$ in the sequel). In the effective lagrangian, $g_{\alpha \beta}$ normalizes the kinetic term of the $E_6$-charged matter fields that are associated to the moduli, while $W_{\alpha \beta \gamma}$ give rise to the Yukawa couplings. The recent trend for the computation of such physical low-energy parameters has been to avoid the notorious difficulty of evaluating correlation functions on the underlying $N = 2$ superconformal field theory, but rather use quite powerful techniques of algebraic geometry and topological field theory [7–26]. The first step in this direction was taken in [7], where it was shown that the couplings could be obtained from the solution of a certain fourth-order linear differential equation of fuchsian type, later identified as a particular example of “Picard–Fuchs equations” obeyed by the periods of the holomorphic threeform $\Omega$ that exists on any Calabi–Yau threefold [15–27]. Similar differential equations can also be derived from mere consistency considerations within the framework of topological Landau–Ginzburg theories [18, 19], where they are easily written down by an iterative algorithm on the superpotential [15, 16]. Later, it was realized that these differential equations are nothing but another way of expressing the geometrical structure of special geometry [12–14, 22]. Indeed, for given $W_{\alpha \beta \gamma}$, Eq. (1) can be viewed as a covariant differential equation for the Kähler potential. Its general solution can be expressed [14, 12] in terms of the $2n + 2$ holomorphic components of the period vector of the three-form $\Omega$ [28, 29, 13], $V_{\gamma}(z) = \int_{\gamma} \Omega$ ($\gamma$ is a given basis of homology cycles), which thus encodes much of the physically relevant data.

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Indeed, one can show [22] that the constraint (1) can be written as a system of coupled fourth-order differential equations on the period vector,

$$D_\alpha D_\beta (W^{-1})^{\gamma \rho \sigma} D_\gamma D_\sigma V(z) = 0$$

(2)

or, equivalently, as $n$ coupled first-order matrix equations

$$(\partial_\alpha - A_\alpha)V = 0,$$

(3)

where $V(z)$ appears as first component of the $(2n + 2) \times (2n + 2)$ period matrix $V$. The characteristic signature of special geometry is that $A_\alpha$ is an $\text{Sp}(2n + 2)$ connection [12, 14]. This gauge invariance can be exploited to eliminate all the non-holomorphic components that in principle appear in Eqs. (1), (2), or (3) in the context of special geometry, and to work with purely holomorphic objects, as is typical of the Picard–Fuchs equations obtained in topological field theories or from algebraic methods. More precisely, in the Kähler and reparametrization covariant derivative $D_\alpha$ in (2), the Christoffel as well as the Kähler-connection naturally split into the sum of two terms [21, 22]. One of them is non-holomorphic and transforms as a tensor whereas the other term is holomorphic and transforms like a connection. Furthermore, the holomorphic pieces of these connections are flat and vanish in “special coordinates” $t^a(z) (a = 1, \ldots, n)$. A similar situation holds in topological Landau-Ginzburg models where the flat connection can be identified with the Gauss–Manin connection [30, 20, 16].

Analogously in (3), using the $\text{Sp}(2n + 2)$ gauge invariance, one can split the connection $A$ into two parts,

$$A_\alpha = \Pi_\alpha + C_\alpha,$$

(4)

where $\Pi_\alpha$ contains the flat connection of TFT’s which vanish in special coordinates, while $C_\alpha$ are the structure constants of a $2n + 2$-dimensional chiral ring $\mathcal{R}^{(3)}$ [22]; they are such that

$$[C_\alpha, C_\beta] = 0, \quad C_\alpha C_\beta C_\gamma C_\delta = 0, \quad C_\alpha C_\beta C_\gamma = W_{\alpha \beta \gamma} E,$$

(5)

where the matrix $E$ has all elements zero but the upper right corner equal to 1.

On the other hand, the moduli space $\mathcal{M}$ of generic string models exhibits non-trivial global properties which are a consequence of its group of discrete isometries $\Gamma$, referred to as the “target space duality” group [31]. The full symmetry structure of the moduli space is then given by $\mathcal{M}/\Gamma$. Duality symmetry, which originates from the fact that the string is an extended object, quite generally describes quantum symmetries of the low-energy effective action. The simplest example is the celebrated $R \rightarrow 1/R$ symmetry of circle compactifications of the bosonic string, which leaves invariant the spectrum upon interchanging winding and momentum modes. This generalizes to an $O(d, d; \mathbb{Z})$ symmetry for toroidal orbifolds. For Calabi–Yau compactifications, or, in a broader sense, for $(2, 2)$ c = 9 superconformal field theories, the mathematical problem of determining the duality symmetry group $\Gamma$ is quite tractable when the moduli space is one dimensional [17], but becomes in general very complex for two or more variables (as (2) become partial differential equations). Only recently it has been tackled for a few two-dimensional examples [1, 24].

An important observation is that the duality group can be inferred from the monodromy of the solutions of the Picard–Fuchs equations (2) or (3). In virtue of their peculiar structure in special coordinates, there is at least one important piece of information that can be gained quite generally for any number of variables: the subgroup of the duality group corresponding to translations

$$t^a \rightarrow t^a + n^a, \quad n^a \in \mathbb{Z}^n,$$

(6)

can be reconstructed from the intersection numbers of the Calabi–Yau manifold [32, 27].

An efficient method for computing the full duality group $\Gamma$ has been recently elaborated in [1]. We exemplify it here through the Calabi–Yau manifold described by a two-parameter deformation of the quintic polynomial immersed in $\mathbb{C}P(4)$,

$$\mathcal{W} = \frac{1}{5}(y_1^5 + \cdots + y_5^5) - a y_1^2 y_2^3 - b y_4^2 y_5^3.$$

(7)

The method is based on algebro-geometric techniques which were previously developed and applied to the study of the monodromy group of Feynman integrals [33, 34], and will be described in the next section.