THE QUANTUM SYMMETRY OF RATIONAL FIELD THEORIES

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The quantum symmetry of a rational quantum field theory is a finite-dimensional multi-matrix algebra. Its representation category, which determines the fusion rules and braid group representations of superselection sectors, is a braided monoidal C*-category. Various properties of such algebraic structures are described, and some ideas concerning the classification programme are outlined.

1. SYMMETRIES AND OBSERVABLES

The notion of symmetry is one of the most fundamental concepts in science. An internal symmetry leaves observable quantities invariant. It only changes the particular manner in which a physical system is described and correspondingly represents a certain amount of redundancy. Therefore it is often a good idea to eliminate internal symmetries so as to arrive at a description in terms of observables only. But there are also various situations where, on the contrary, the redundancy is highly welcome. For instance, often the use of redundant variables leads to simpler coordinates and thereby facilitates a perturbative treatment. Another example arises in the context of what I will call quantum symmetry. To give a first impression of what is meant by this term, let me specialize to the specific case of conformal field theory: By a quantum symmetry of a two-dimensional conformal field theory I mean an internal symmetry whose representation theory reproduces the basis independent contents of the operator product algebra, i.e., the fusion rules, and which is compatible with the duality properties of chiral blocks. A model independent characterization of the notion of quantum symmetry will be given at the beginning of Section 3.

While conformal invariance is operational in the latter characterization, it is far from being an essential ingredient. Rather, the type of quantum symmetries encountered in conformal field theory turns out to be relevant to relativistic quantum field theory in general. To make this remark more concrete, let me consider a specific axiomatic formulation of quantum field theory, namely the algebraic theory of superselection sectors [1-4], to which I will refer in the sequel by the term algebraic field theory for short. Within the framework of algebraic field theory, a theory is described in terms of nets of von Neumann algebras \( \mathcal{A}(\mathcal{O}) \) which are indexed by specific open subsets \( \mathcal{O} \) of Minkowski space-time. Consequently, the proof of various results from algebraic field theory on which I will rely below requires the mathematics of von Neumann algebras and subfactors [5-8]; however, the essence of these results is in all relevant cases rather plausible already from a more heuristic point of view. My perspective will therefore be to take these results for granted and investigate their implications for quantum symmetry.

Classical physics can be described via a configuration space \( X \) endowed with a measure \( \mu \); the dynamics is then described in terms of the elements of the algebra \( L^\infty(X, \mu) \) of bounded measurable functions on \( X \). In quantum physics, one trades the configuration space, respectively the commutative algebras of functions on it, for non-commutative \( * \)-algebras of operators. The analogue of the measures \( \mu \) are states (normalized positive linear forms) over the algebra. A state induces a scalar product on the algebra, and the completion of the algebra with respect to the associated norm is a separable Hilbert space \( \mathcal{H} \) on which the algebra acts by multiplication. Thus, physical states correspond to the vectors \( \psi \) of a Hilbert space \( \mathcal{H} \); the analogue of \( L^\infty(X, \mu) \) is then the field algebra \( \mathcal{F} \) which is a subalgebra of the algebra \( B(\mathcal{H}) \) of bounded operators on \( \mathcal{H} \). The vectors of \( \mathcal{H} \) can be thought of as being created by acting with field operators \( f \in \mathcal{F} \) on a vacuum vector \( \Omega \). In this context, the observables are the (self-adjoint) elements of a \( * \)-subalgebra \( \mathcal{A} \) of \( \mathcal{F} \). Any measuring apparatus is contained in a bounded region \( \mathcal{O} \) of space-time, and hence there must exist local observable algebras \( \mathcal{A}(\mathcal{O}) \) of observables measurable in \( \mathcal{O} \). The local observable algebras associated with causally disconnected regions commute among one another (Einstein causality). The total observable algebra \( \mathcal{A} \subseteq \mathcal{F} \) is the quasilocal \( C^* \)-algebra \( \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O}) \).

A distinctive feature of quantum symmetries in low-dimensional field theory is that they correspond to finite-dimensional algebras. Recall that in classical physics the (finite) symmetry transformations correspond to the elements of a group \( G \), or rather to the associated group algebra \( CG \). If \( G \) is infinite, say a Lie group, then \( CG \) is infinite-dimensional. As it turns out, this situation prevails in quantum theory as long as the dimensionality of space-time is large enough.
In contrast, in low-dimensional quantum field theory there are special systems, the so-called rational field theories, for
which the quantum symmetry is a finite-dimensional algebra which is generically not a group algebra. Heuristically,
the possibility of having more general structures than group algebras can be understood by investigating the question
of what happens to the group of symmetries in the course of quantization of a classical system. Namely, elements of the
group $G$ can be viewed as acting on the points of the configuration space. Since in quantum physics the configuration
space is no longer present, the group elements are not needed any more either.

2. SUPERSELECTION SECTORS

In algebraic field theory the observables are taken as the basic objects. Because of Einstein causality, observables
must commute at space-like separations, or, in other words, the statistics of observables is bosonic. One of the chal-
lenges of algebraic quantum field theory is the investigation of the possible statistics (i.e., the behavior with respect to
permutations) of non-observable fields. The action of the field algebra $\mathcal{F}$ on the vacuum vector $\Omega$ is generically reducible;
correspondingly, there are sub-Hilbert spaces of the Hilbert space $\mathcal{H}$ that are orthogonal to each other. These subspaces
are called superselection sectors; denoting them by $\mathcal{H}_\alpha$, one has the sector decomposition
\[
\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha.
\] (1)

Observables act within the sectors. (This implies that the relative phases of vectors belonging to different subspaces
$\mathcal{H}_\alpha$ cannot be observed. Hence, in contrast to the situation in quantum mechanics, the superposition principle is not
valid universally, but holds only within sectors.) Thus, each sector $\mathcal{H}_\alpha$ carries a representation $\pi_\alpha$ of $\mathcal{A}$. The general
representation theory of $\mathcal{A}$ is rather wild, but fortunately in quantum field theory only a restricted class of representa-
tions is relevant. The precise requirements that these physical representations must meet depend on the specific axiomatic
framework; in the present context, the relevant notion [1-4] is the one of DHR (Doplicher–Haag–Roberts)-representa-
tions. Recall that a vacuum representation $\pi_0$ of $\mathcal{A}$ is a positive-energy representation for which the associated Hilbert
space $\mathcal{H}_0$ contains a unique (up to normalization) vector $\Omega$, called the vacuum vector, which is cyclic and separating for
$\mathcal{H}$ and is invariant under the relevant space-time symmetry transformations, in particular under translations. Then by
definition a DHR representation is a representation which is isomorphic to $\pi_0$ outside some bounded region, or, in other
words, is a ‘local excitation of a vacuum representation’. It will also be assumed below that the index of inclusion of the
algebra $\pi_\alpha(\mathcal{A}(O))$ in the commutant of $\pi_\alpha(\mathcal{A}(O'))$ ($O'$ denotes the space-like complement of $O$) with respect to $B(\mathcal{H})$
is finite. DHR representations with this property are said to have finite statistics.

In more technical terms, DHR representations $\pi_\alpha$ are characterized by being unitarily equivalent to $\pi_0$ in the sense that
\[
\pi_\alpha \cong \pi_0 \circ \rho_\alpha ,
\] (2)
where $\rho_\alpha : \mathcal{A}(O) \to \mathcal{A}(O)$ are endomorphisms of the local algebras $\mathcal{A}(O)$ which act as the identity map on the von
Neumann algebra generated by the local observables in $O'$. (This implies that the algebras generated by the local ob-
servables in different sectors are isomorphic, and the sectors can only be distinguished by global quantities, which are
referred to as superselection charges.)

Several properties of DHR representations with finite statistics are relevant to the investigation of quantum sym-
metry. They are given in the following list.

\begin{itemize}
  \item The representations are irreducible, and they appear in the sector decomposition (1) with finite multiplicity. Ac-
cordingly (1) can be rewritten as
  \[
  \mathcal{H} = \bigoplus_p (\mathcal{H}_p \times \mathbb{C}^{n_p}) ,
  \] (3)
  where the representations $\pi_p$ corresponding to the Hilbert spaces $\mathcal{H}_p$ are pairwise inequivalent, and where $n_p < \infty$
are non-negative integers.
  The vacuum sector is non-degenerate, $n_0 = 1$.
  \item It is possible to define a tensor product of (unitary equivalence classes of) the representations $\pi_p$ of $\mathcal{A}$; namely,
  \[
  \pi_p \times \pi_q \cong \pi_0 \circ \rho_p \circ \rho_q .
  \] (4)
  \item The product of sectors is completely reducible. Thus,
  \[
  \pi_p \times \pi_q \cong \bigoplus_r (\mathbb{C}^{N_{pq}^r} \times \pi_r)
  \] (5)
  for some non-negative integers $N_{pq}^r$. I will refer to these integers as fusion rule coefficients. One has $\pi_0 \times \pi_q \cong \pi_q$, i.e., $N_{0q}^r = \delta_0^r$.
\end{itemize}