The so-called homogeneous version of differential renormalization recently proposed by V. A. Smirnov and the author is formulated in the momentum representation. The mechanism of subtractions is revealed. New possibilities are exhibited.

1. INTRODUCTION

The recently proposed new version [3] of differential renormalization [1, 2] was based on the differential operator

\[ \tilde{S} = \frac{1}{2} \sum_{i=1}^{n} \sum_{\nu=0}^{3} \frac{\partial}{\partial x_{i\nu}}(x_{i\nu} - \bar{x}) - \frac{1}{2} \sum_{l=1}^{L} m_{l} \frac{\partial}{\partial m_{l}} \]

(1)

rather than on the Laplacian of [1] and [2]. Here \( x_{i} \) are coordinates of vertices (their total number is \( n \)) of a diagram \( \Gamma \), subjected to renormalization, and \( \bar{x} \) is the coordinate of the "center of vertices" of \( \Gamma \), namely,

\[ \bar{x} = \frac{1}{n} \sum x_{i}. \]

(2)

The parameters \( m_{l} \) are masses corresponding to lines of the diagram. The unrenormalized amplitude \( \Pi \) of the diagram is the product of propagators in coordinate space:

\[ \Pi(x_{1}, \ldots, x_{n}) = \prod_{i<j}[G(x_{i} - x_{j})]^{L_{ij}} \]

(3)

(here \( L_{ij} \) is the number of lines that connect vertices \( i \) and \( j \)). Similar notations are used for Feynman amplitudes of subgraphs \( \gamma_{1}, \gamma_{2}, \ldots, \gamma_{k} \), viz., \( \Pi_{1}, \Pi_{2}, \ldots, \Pi_{k} \). Usually, for the sake of brevity, we omit the space-time indices in \( \tilde{S} \) so that, e.g., for a massless theory, the operator \( \tilde{S} \) takes the form

\[ \tilde{S} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial x_{i}}(x_{i} - \bar{x}). \]

(4)

Another important object is the operator \( \tilde{D} \) related to the generator of dilatations:

\[ \tilde{D} = \frac{1}{2} \sum_{i=1}^{n}(x_{i} - \bar{x}) \frac{\partial}{\partial x_{i}} \equiv \tilde{S} - 2n + 2. \]

(5)

Analogous quantities are introduced for subgraphs \( \gamma_{1}, \ldots, \gamma_{k} \) of \( \Gamma \). Thus we deal with objects like \( \tilde{S}_{k}, \tilde{D}_{k}, L_{k} \), etc., given by similar formulae but with summation only over elements of a subgraph \( \gamma_{k} \). We call this variant of differential renormalization homogeneous in order to distinguish it from the initial version [1, 2] based on the Laplacian (the reason for the name is that our procedure makes intensive use of the homogeneity properties of field propagators). We emphasize once again that contrary to [1, 2] our procedure is readily applicable to diagrams with non-zero spins and non-zero masses.

Now we recall the mechanism of renormalization. For a primitively logarithmically divergent diagram \( \Gamma \) whose unrenormalized amplitude is \( \Pi(x) \) one constructs a function \( F(x) \) which satisfies the equation

\[ \tilde{S} F(x) = \Pi(x). \]
The singularities of \( \Pi(x) \) are localized of course on manifolds where \( u_i = x_i - \bar{x} \) are zero. If these pole singularities are non-integrable, one encounters ultraviolet divergences, so the relation (6) makes sense only beyond the above-mentioned manifolds. As mentioned in our previous publications, \( F \) acquires only logarithms of \( \Pi \) as additional singularities in comparison to \( \Pi \). Every term of \( S \) contains preliminary multiplication by \( u_i \) and a subsequent differentiation over \( x_i \). If the ultraviolet divergence is logarithmic, then the function \( (x_i - \bar{x}) F \) happens to be locally integrable and therefore defines some "good" distribution on the overall test-function space. If so, the term \( \frac{\partial}{\partial x_i} (x_i - \bar{x}) F \) is also well defined (remember that derivatives of a distribution are always well-defined). This is the principal way of constructing the desired extension of the unrenormalized Feynman amplitude to the renormalized one. It was shown in [3] that for a primitively logarithmically divergent massless diagram \( F = -\frac{1}{L} \ln \Pi \) so that the renormalized amplitude \( R \Pi \) is

\[
R \Pi = -\frac{1}{L} \hat{S} \Pi \ln \Pi.
\]

(7)

where \( L \) is the number of lines of the graph considered, and \( \log \) might contain a scale factor \( \mu \) which in (7) is supposed to be \( \mu = 1 \). Analogously, for a diagram with two logarithmic divergences \( \gamma_1 \) and \( \Gamma, \gamma_1 \subset \Gamma \) and for a diagram with three divergences \( \gamma_1, \gamma_2, \) and \( \Gamma, \gamma_1 \subset \gamma_2 \subset \Gamma \) one gets

\[
R \Pi = -\frac{1}{L_1} \hat{S}_1 \Pi \ln \Pi_1 - \frac{1}{2L_1^2} \hat{S}_1 \Pi_1 \Pi_2^2 (\ln \Pi_1)^2 \hat{D}_1 \Pi_1 \Pi_2
\]

(8)

and

\[
R \Pi = -\frac{1}{L_1} \hat{S}_1 \Pi \ln \Pi_1 - \frac{1}{2L_1^2} \hat{S}_1 \Pi_1 \Pi_2^2 (\ln \Pi_1)^2 \hat{D}_1 \Pi_1 \Pi_2
\]

- \frac{1}{2L_1^2} \hat{S}_1 \Pi_1 (\ln \Pi_1)^2 \left( \hat{D}_1 \Pi_1 \Pi_2 \right)

- \frac{1}{6L_1^3} \hat{S}_1 \Pi_1 (\ln \Pi_1)^3 \left( \hat{D}_1 \Pi_1 \Pi_2 \right).

(9)

Analogous explicit expressions have been obtained as well for diagrams with disjoint subdivergences and for some classes of diagrams with overlapping subdivergences. The general recurrence relation for constructing an explicit expression for an arbitrary diagram has been presented. If some internal masses of a theory are non-zero one can apply the same formulae (7), (8), and (9). Still, even for non-zero masses and higher spin diagrams, it is more convenient to use massless scalar amplitudes \( \Pi^{(0)} \) under the signs of \( \ln \), in the corresponding relations. Since the massless scalar propagator \( G(u) \) is proportional to \( u^{-2} \) one finds

\[
\ln \Pi^{(0)} = \sum_i \ln u_i^{-2},
\]

where summation goes over all lines of a diagram and \( u_i \) is the argument of the \( i \)-th propagator. Thus, in its essentials, the renormalization procedure amounts to two steps: a) one chooses a line in the diagram and changes it into, say, a "thick" line, i.e., modifies the corresponding propagator by multiplying it by \( \ln u_i \); b) then one applies the operator \( \hat{S} \) (we do not mention here manipulations of secondary importance: symmetrization over lines and combinatorial complications due to non-trivial configuration of divergences). The general case of an arbitrary divergence degree \( \omega \) can be considered similarly based on the differential operator \( \hat{S}(\omega) \):

\[
\hat{S}(\omega) = \hat{S} \left( \hat{S} + \frac{1}{2} \right) \left( \hat{S} + 1 \right) \ldots \left( \hat{S} + \frac{\omega}{2} \right).
\]

(10)

There are several possibilities for formulating the renormalization recipe for the case \( \omega > 0 \). One should choose a specific variant in order to preserve the internal symmetry of a model. We'll discuss this subject elsewhere.

The aim of the present paper is to reformulate the homogeneous differential renormalization procedure in terms of the momentum representation. There are three strong motivations for this. The first one is that most of the calculations are usually performed in momentum space. The second one is that counterterms introduced by any renormalization are local in the coordinate space. This makes it necessary to use the very careful language of distributions, test functions, etc. when analyzing, say, the symmetry properties of renormalized quantities. Contrary to that, counterterms are polynomials in the momentum space. So they are more "noticeable" and can be treated in a less sophisticated way. The third motivation is that "visualizing" the mechanism of subtractions in the momentum representation will help us to generalize the possible procedure considerably.