CAUCHY PROBLEM FOR A SEMILINEAR WAVE EQUATION.

III

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There is given a revision of the formulation and the proof of the theorem regarding the global unique solvability in the class of weak (energy) solutions of the Cauchy problem, for a second-order semilinear pseudodifferential hyperbolic equation on a smooth Riemannian manifold (of dimension \( n \geq 3 \)) without boundary (see the author's previous paper in Zap. Nauchn. Sem. LOMI, Vol. 182, 1990). Under natural additional assumptions it is proved that if the initial data \( u(0, x), \partial_t u(0, x) \) are smoother:

\[
\begin{align*}
    u(0) & \in H^{s+1}, \\
    \partial_t u(0) & \in H^s, \quad 0 < s \leq 2,
\end{align*}
\]

then also the weak solution is smoother: \( u \in C([0,T] \to H^{s+1}), \partial_t u \in C([0,T] \to H^s) \).

Introduction

0.1. We consider the semilinear wave equation

\[
\ddot{u}(t,x) - \Delta u(t,x) + f(u(t,x)) = 0, \tag{0.1}
\]

where \( x \in \mathbb{R}^n \), \( \Delta \) is the Laplace operator in \( \mathbb{R}^n \), \( f(\cdot) \) is a scalar function, a dot above \( u \) denotes derivative with respect to the time \( t \).

In recent years, the question of the timewise global unique solvability of the Cauchy problem for multidimensional (\( n \geq 3 \)) equations of the form (0.1), with initial data

\[
\begin{align*}
    u(0, x) & = \varphi(x), \\
    \dot{u}(0, x) & = \psi(x),
\end{align*}
\]

from one or another functional space (without any assumption regarding the smallness of \( \varphi \) and \( \psi \)) has been investigated by several authors (see, for example, [4--8, 13, 16--20, 25]). Basically, one has investigated either the weak solutions [7, 8, 13], corresponding to the initial data \( \varphi, \psi \) from the spaces \( L^2(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \), respectively, or the strong solutions [4--6, 16--19], corresponding to smoother initial data: \( \varphi \in H^s(\mathbb{R}^n), \psi \in H^s(\mathbb{R}^n) \). The rigorous definition of the
solution of the problem (0.1), (0.2) will be given later below (see Subsection 0.4) and now we only mention that a weak
solution \( u \) on the interval \([0, T]\) has the property
\[
\{ u, \dot{u} \} \in L_\infty([0, T] \to H^1 \times L_2),
\]  
while for a strong solution we have
\[
\{ u, \dot{u} \} \in L_\infty([0, T] \to H^3 \times H^1).
\]  

In the above mentioned investigations, on the function \( f \) (the nonlinearity) one has imposed one or another condition,
the most essential of which has been a restriction on the order of growth of \( |f(u)| \) for \( |u| \to \infty \). For the illustration of
the existing results we turn to the case of pure power nonlinearity:
\[
f(u) = |u|^{p-1}u.
\]  
Under these assumptions, the problem (0.1), (0.2) has for any \( \varphi \in H^1 \) and \( \psi \in L_2 \) a weak solution, global with respect to \( t \),
if \( p \) satisfies the natural restrictions
\[
l < p < \frac{n+2}{n-4},
\]  
guaranteeing the finiteness of the energy
\[
E(u; t) = \int \left[ \frac{1}{2} |\dot{u}(t, x)|^2 + \frac{1}{2} |u_x(t, x)|^2 + \frac{1}{p+1} |u(t, x)|^{p+1} \right] dx
\]  
for \( t = 0 \) (\( \forall \psi \in H^1, \forall \varphi \in L_2 \)) and for a. a. \( t > 0 \) (see [21]). The uniqueness of the weak solution in the case \( f \leq \varphi / (n-2) \) can
be verified in a sufficiently easy manner (see, for example, [15], Chap. 1). Incomparably more complicated is the proof of the
uniqueness in the case of arbitrary \( n \geq 3 \) and \( p \) from the interval
\[
l < p < \frac{n+2}{n-4},
\]  
presented only recently by Ginibre and Velo [7, 8].

At the investigation of the Cauchy problem (0.1), (0.2) in the class of strong solutions, the fundamental problem turns
out to be not so much the uniqueness theorem as the existence theorem. The fact is that if \( \varphi \in H^k \), \( \psi \in H^1 \) and in the interval
\([0, T]\) there exists a solution \( u \) of the problem (0.1), (0.2) with the property (0.4), then, in the class of such solutions, \( u \) will
be unique as soon as \( p \) satisfies, for example, the condition \( 1 \leq p \leq n/(n-4) \). This can be proved easily with the aid of the
usual energy estimate for the linear nonhomogeneous wave equation if we take into account (0.4) and we make use of the
Sobolev imbedding theorem.

For \( f \leq \varphi / (n-2) \) the existence of a global strong solution is established by entirely elementary means [9], but the
problem becomes significantly more complicated if \( f > \varphi / (n-2) \).

In Pecher's papers [17-19] the theorem of the existence of a global strong solution is proved for arbitrary \( p \) from the
interval (0.7) in the case of the dimensions \( 3 \leq n \leq 9 \) and for
\[
l < \frac{n+2}{n-4} - \varepsilon_n
\]  
with some \( \varepsilon_n > 0 \) in the case \( n \geq 10 \). A similar result under conditions on \( f(u) \) that are slightly weaker than those of Pecher
and for more general equations has been obtained in [4, 5]. It should be mentioned that the conditions on \( f \) in [4-8, 17-19]
include an assumption regarding a sufficient smoothness of \( f \); at least \( f \in C^2 \) and, therefore, strictly speaking, the mentioned
investigations do not encompass the case of a pure power nonlinearity (0.5) for \( n > 6 \) and \( f \leq \varphi / (n-2) \). This case has been
considered specially in [16], where it is shown that for \( 6 \leq n \leq 10 \) the exponent \( p \) can be arbitrary from the interval (0.7).
A recent investigation of Brenner [6] is devoted to the proof of the existence of strong solutions under the assumption (0.7)
for all \( n \geq 3 \); however, apparently, the key estimate (1.2) of [6] (p. 48) is false.

Parallel with the strong solutions of the problem (0.1), (0.2), in [4, 5, 16-19] one has considered also smoother
solutions \( \{ u \in L_\infty([0, T] \to H^{5+\delta}) \} \), but under stronger restrictions on \( f \). Thus, for example, Theorem III.3 of [5] asserts that
if \( f \in C^6 \), \( \int_0^\infty f(u) du > 0 \), \( \forall u > 0 \), and for derivatives \( f^{(j)} \) we have inequalities \( |f^{(j)}(u)| \leq c(|u|^{p+1}) \), \( |f^{(j)}(u)| \leq c(|u|^{p+1})^{1/2} \) with.