RELAXATION OF A LIQUID LAYER UNDER THE ACTION OF CAPILLARY FORCES

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The theory of creeping motion is used to study the relaxation of an infinite viscous fluid layer (membrane) of nonuniform thickness. The propagation of boundary perturbations in a semi-infinite layer under the action of surface-tension forces is also considered. The layer has at least one common boundary with a gas. It is found that relaxation processes of an infinite layer or the propagation of boundary perturbations inside a bounded layer are nonmonotonic, and that wave-like surface perturbations always arise. Relaxation times are determined. Maximum distances are found over which separate regions of the layer can affect each other.

1. Fundamental Equations. It is assumed that the thickness $h$ of the viscous fluid layer varies over distances $l$ such that $l \gg h$, i.e., $dh/dx \ll 1$ ($x$ is the coordinate in the direction of the layer). We know [1] that the equations of hydrodynamic lubrication theory are valid when the reduced Reynolds number $R^* \ll 1$ ($R^* = \nu h^2/\eta$, $\nu$ is the velocity along the layer). For small wave-like perturbations, when the variation of thickness $\Delta h \ll h$, this condition is insufficient, since the nonsteady-state term in the Navier-Stokes equation can be large. We must therefore take the more general condition $h^2 \ll \tau^2$, where $\tau$ is the characteristic time for variation of the layer thickness.

The equations of motion and conservation of mass have the form [2]

$$\frac{\partial p}{\partial x} = \rho g + \mu \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial}{\partial x} \left[ h \frac{\partial v}{\partial y} \right] + \frac{\partial h}{\partial t} = 0.$$ \hspace{1cm} (1.1)

Here $y$ is the coordinate across the layer; $y = 0, y = h$ are the coordinates of the surfaces bounding the layer; $g$ is the mass force.

In the case of a membrane situated on a solid surface, we can assume that a constant shear stress $F$, applied externally, acts on the free surface of the membrane. Consequently $\mu \frac{\partial v}{\partial y} = F$ for $y = h$. Clearly $v = 0$ and $y = 0$ also. These boundary conditions and the first of Eqs. (1.1) are satisfied by

$$2\nu \frac{\partial y}{\partial x} = (\partial p/\partial x - \rho g) (y^2 - 2yh) + 2Fy.$$ \hspace{1cm} (1.2)

If the second equation of (1.1) is taken into account we have

$$\frac{\partial}{\partial x} \left[ \frac{3h}{2\eta} \left( \frac{\partial p}{\partial x} - \frac{3}{2h} F - \rho g \right) \right] = \frac{\partial h}{\partial t}.$$ \hspace{1cm} (1.2)

It is known [2] that the boundary condition at the free surface of a fluid can coincide with the boundary condition of a solid body if substances with surface activity are present. In what follows, membranes with this type of boundary condition are referred to as stabilized membranes. It is not difficult to obtain an equation similar to (1.2) for a stabilized layer, if we allow for the fact that the layer suffers only sym-
metric deformations relative to the center, because the pressure is constant over the cross section and there are surface-tension forces acting. This equation has the form

\[
\frac{\partial}{\partial t} \left[ \frac{h}{12\mu} \left( \frac{\partial p}{\partial x} - \rho g \right) \right] = \frac{\partial h}{\partial t} .
\]  

(1.3)

This equation is treated in [3] for the case in which \( \partial p/\partial x = 0 \). If it is assumed that the gas pressure at the free surface is constant, the following expressions may be written down for the pressure inside the membrane:

\[
(p - p_0) = \sigma \frac{\partial h}{\partial x}, \quad (p - p_0)_2 = \frac{1}{12} \sigma \frac{\partial h}{\partial x}^2 .
\]  

(1.4)

Here \( \sigma \) is the surface-tension coefficient; the subscript 1 refers to the membrane on the solid surface; the subscript 2 refers to a stabilized membrane having a boundary with a gas only.

If one of the equations (1.4) is inserted in (1.2) or (1.3) and the result linearized, the following equation is obtained:

\[
\frac{\partial h}{\partial t} = - a \frac{\partial h}{\partial x}^2 - b \frac{\partial h}{\partial x} .
\]  

(1.5)

For a nonstabilized membrane on a solid surface

\[
b \mu = \frac{1}{12} \sigma \rho F + \rho g h^2, \quad 3 \mu a = \sigma h^3 .
\]

For a stabilized membrane having a boundary with a gas only

\[
4 b \mu = \rho g h^2, \quad 24 \mu a = \sigma h^3 .
\]

If one of the boundaries is a solid body then the coefficient \( b \) remains the same, while the coefficient \( a \) is doubled.

2. An Infinite Membrane. The Cauchy problem can be correctly formulated for equation (1.5) if \( a > 0 \), as can be seen from what follows. For \( a < 0 \) the formulation is incorrect. Since \( a > 0 \) always for a membrane, the problem can be formulated with the initial condition

\[
h = h_0(x) \text{ for } t = 0, -\infty < x < +\infty .
\]  

(2.1)

A Laplace transform with respect to time and a Fourier transform with respect to the space coordinate can be used in order to solve Eq. (1.5) with the initial condition (2.1):

\[
h(k, p) = \int_0^\infty dt \int_{-\infty}^{\infty} h(x, t) e^{-pt+ikx} dx .
\]  

(2.2)

Equation (1.5) then gives

\[
ph(k, p) - h_0(k) = - (ak^4 + bik)h(k, p)
\]

where \( h_0(k) \) is the Fourier transform of the function \( h_0(x) \). The Fourier transform of the function \( h(x, t) \) can then be found easily:

\[
h(k, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{pt} h(k, p) dp = h_0(k) e^{-a(k^4 + bik)t} .
\]  

(2.3)

When the inverse Fourier transform is taken and the convolution theorem used, \( h \) can be expressed in terms of \( h_0 \) with the help of the Green's function:

\[
h(x, t) = \int_{-\infty}^{\infty} h_0(\xi - bt) G(x - \xi, t) d\xi .
\]  

(2.4)