THE INTEGRAL REPRESENTATION OF FRACTIONALLY EXPONENTIAL FUNCTIONS AND THEIR APPLICATION TO DYNAMIC PROBLEMS OF LINEAR VISCO-ELASTICITY

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Fractionally exponential functions are written in the integral form and distribution functions with an Abelian singularity are obtained for the corresponding relaxation and retardation spectra. A principle is stated, defining the dynamic problems for which weakly singular functions can be used as the kernels of the integral operators. A one-dimensional sound wave traveling in a seminfinite visco-elastic medium is considered. The generalized exponential functions of fractional order, proposed by Yu. N. Rabotnov [1, 2] as the kernels of Boltzmann-Volterra integral relations, have found wide applications in theory of linear visco-elasticity. This is explained partly by the great mathematical flexibility of the F-operators when applying the Volterra principle to the solution of elastically hereditary problems and partly by the fact that almost all weakly singular kernels possessing an Abelian singularity are connected in some way or other with the F-functions. For example, the resolvent of the elementary weakly singular Abelian kernel is an F-function. The product of an exponential function with an Abelian kernel represents a particular case of the product of two F-functions with different fractional parameters, while the resolvent of such a kernel is the product of an exponential function with an F-function [3, 4]. Since the e-functions are defined by slowly convergent series, their various asymptotic forms [2, 5-8] are commonly used in practical calculations. The theory of F-functions can be developed further in the context of their integral representations, which enables a more exact physical interpretation to be given to their parameters and on occasion simplifies computational operations.

1. The most general definition of F_y-functions is given in [1]; while the same approach will be used here, the working will be performed in Laplace space and different notation is introduced.

The following relations between the stress \( \sigma \) and deformation \( \varepsilon \) are taken as fundamental:

\[
\begin{align*}
\sigma &= E_\infty \left[ \varepsilon(t) - \nu_e \int_0^\infty R(t') \varepsilon(t - t') dt' \right], \\
\nu_e &= (E_\infty - E_0) E_\infty^{-1} \\
\varepsilon &= J_\infty \left[ \sigma(t) - \nu_\sigma \int_0^\infty K(t') \sigma(t - t') dt' \right], \\
\nu_\sigma &= (J_0 - J_\infty) J_\infty^{-1}.
\end{align*}
\]

Fig. 1
Here $E_{\infty} = J_{\infty}^{-1}$, $E_0 = J_0^{-1}$ are respectively the nonrelaxation and relaxation values of the elastic modulus and pliability, while $R(t)$ and $K(t)$ are the relaxation and after-effect kernels, which are expressible in terms of the distributions $A_1(\tau, \tau_e)$ and $B_1(\tau, \tau_\sigma)$ of the relaxation time $\tau_e$ and retardation time $\tau_\sigma$ respectively

$$R(t) = \int_0^\infty \tau^{-1} A_1(\tau, \tau_e) e^{-t/\tau_e} d\tau, \quad K(t) = \int_0^\infty \tau^{-1} B_1(\tau, \tau_\sigma) e^{-t/\tau_\sigma} d\tau. \quad (1.2)$$

The expression connecting the Laplace transforms $R_\lambda(p)$ and $K_\lambda(p)$ of the relaxation and after-effect kernels is

$$E_{\infty} - E_0 = E_{\infty} R_\lambda(p) - E_0 K_\lambda(p). \quad (1.3)$$

The simplest hereditary function possessing an integrable singularity is the Abel kernel, which characterizes the unsteady process. It can therefore be used meaningfully as the after-effect kernel for describing unsteady creep [9]. But formally, there are two possibilities

$$K(t) = t^{-1} / \Gamma(\gamma) \tau_\sigma^{\gamma}, \quad R(t) = t^{-1} / \Gamma(\gamma) \tau_e^{\gamma}, \quad E_0 \tau_e^{\gamma} = E_{\infty} \tau_\sigma^{\gamma}. \quad (1.4)$$

Here $\Gamma(\gamma)$ is the gamma function, and $\gamma$ its parameter of singularity ($0 < \gamma \leq 1$). The following is then obtained from (1.3) for the corresponding resolvents in Laplace space:

$$R_\lambda(p) = \nu_{\gamma^{-1}} [\nu^{-1}(\nu \tau_\sigma)^{\gamma} + 1]^{-1}, \quad K_\lambda(p) = \nu_{\gamma^{-1}} [\nu^{-1}(\nu \tau_e)^{\gamma} - 1]^{-1}. \quad (1.5)$$

Here and below, we take $\tau = \tau_e$ if $\nu = \nu_e$, and $\tau = \tau_\sigma$ if $\nu = \nu_\sigma$, where $\tau$ and $\nu$ appear without subscripts.

There are two ways of transforming to the space of originals: first, by formal expansion of the right sides of (1.5) in powers of $\nu(\nu \tau_e)^{-\gamma}$, followed by term-by-term passage to the original, and second, by direct application of the Mellin-Fourier inversion formula. In the first case,

$$R(t) = \tau_e^{-\gamma} F_{\gamma}(-\nu, t/\tau), \quad K(t) = \tau_\sigma^{-\gamma} F_{\gamma}(\nu, t/\tau),$$

$$F_{\gamma}(\pm\nu, t/\tau) \equiv t^{-1} \sum_{n=0}^{\infty} \frac{(-\nu)^n (t/\tau)^{\gamma n}}{\Gamma(n+1)}. \quad (1.6)$$

Here $F_{\gamma}$ is the Rabotnov fractionally exponential function [1, 2], which is seen from (1.6) to be defined by either an ordinary or an alternating series.

When the Mellin-Fourier formula is applied directly, an integral form is obtained for the $F_{\gamma}$-function

$$R(t) = \tau_e^{-\gamma} F_{\gamma}(-\nu, t/\tau) = \frac{\nu_{\gamma^{-1}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp(pt) dp}{1 + \nu^{-1}(\nu \tau_e)^{\gamma}}, \quad (1.7)$$

Here the case when the minus sign appears in front of the parameter $\nu$ of the $F_{\gamma}$-function is taken for clarity, since this is the meaningful case as regards applications.

When $\gamma \neq 1$, the singular points of the integrand of (1.7) are the branch points $p = 0$ and $p = \infty$, and the simple poles at the $p$ for which the denominator $(\nu \tau_e)^{\gamma} + \nu$ vanishes. The latter are

$$p_{1,2} = \tau_e^{-\nu_{\gamma^{-1}}} \cos(\pi/\gamma) \pm i \sin(\pi/\gamma). \quad (1.8)$$

The inversion theorem can only be applied to many-valued functions having branch points on the first sheet of the Riemann surface, i.e., with $-\pi = \arg p = \pi$, and the residues at the points $p_{1,2}$ are discounted when evaluating the line integral (1.7). On the other hand, when $\gamma = 1$, the one singular point $p = \nu \tau_e^{-1}$ is a pole of the first order, and the integral is evaluated simply by finding the residue at this point. Taking a contour of integration with a cut along the negative real axis and applying Cauchy’s theorem, we get