SCATTERING BY NONCOMPACT OBSTACLES

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For Schrödinger's equation and for the wave equation one considers scattering by noncompact obstacles, satisfying the illumination condition. One assumes that the Dirichlet boundary condition is satisfied; for obstacles of cone type, one considers also other boundary conditions. One investigates the existence, the isometry, and the completeness of the operators.

1. Introduction. The existence of "direct" wave operators (w.o.) for the Schrödinger operator (and more general operators) at the scattering on noncompact obstacles is established in a relatively simple manner. Significantly more difficult is the question of the completeness of the w.o. Interesting results regarding this have appeared recently in [1], where the w.o. are investigated on the basis of expansion theorems obtained there; one considers the operator $-\Delta + q(x)$ under the condition D (Dirichlet) on the obstacle. Here we present another method of investigation of the w.o. (see the comparison with [1] in Sec. 8). The starting point is formed by considerations of an abstract character, close to the theory of $H$-smooth perturbations of Kato (see also [2, 3]). As opposed to [2, 3], the presented variant of the theory is "asymmetric," which widens the possibilities for applications. It is suitable also for the investigation of second order equations with respect to time. The analytic scheme is based on weighted estimates and on the limiting absorption principle (Sec. 5) for second-order elliptic equations. Such estimates under the condition D are obtained in [4, 5]; we strengthen them, assuming increasing weights. One succeeds to obtain similar estimates for obstacles of conical type also under other boundary conditions. If the estimates are obtained, then the investigation of the w.o. according to the scheme presented here is not too complicated. From other technical tools we mention the use of "cuts" with special properties (Sec. 4).

A detailed proof of the results will be presented elsewhere.

2. Abstract Scheme. Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$; let $E(\Delta)$, $\mathcal{R}(z)$ be its spectral measure and resolvent. The same meaning have $E_0(\Delta)$, $\mathcal{R}_0(z)$ for the operator $H_0 = H^* \in \mathcal{H}$ in the Hilbert space $\mathcal{H}$. By $B(\mathcal{H}, \mathcal{K})$ (by $C(\mathcal{H}, \mathcal{K})$) we denote the collection of continuous (closed, densely defined) linear mappings of $\mathcal{H}$ into $\mathcal{K}$; moreover, $B(\mathcal{H}, \mathcal{K}) = B(\mathcal{K}, \mathcal{H})$ etc. Below $\Delta = [a, b]$ is a bounded interval and the operators $H$, $H_0$ are absolutely continuous on $\Delta$. The term "almost everywhere" (a.e.) refers to the Lebesgue measure. Let $F(\mathcal{R}(z)) \in B(\mathcal{H})$ and assume that for a set $g$ of elements dense in $\mathcal{H}$ for a.e. $\lambda \in \Delta$ there exist the weak limits

$$2\pi i \lim_{t \to 0} \mathcal{R}(\lambda \pm it)q = \mathcal{R}_0(\lambda)q \in L^p(\Delta), 1 \leq p < \infty.$$
Then we shall write \( F \in \Lambda_p(H; \Delta) \). The operator \( T \in C(\mathcal{Y}_p^0, \mathcal{Y}_p) \) is said to be \( \mathcal{H}_0 \)-smooth (see [3]) if

\[
\left\| T \exp(-i t \mathcal{H}_0) f \right\| L^2 \leq c(T) \| f \| L^2, \quad \forall f \in \mathcal{Y}_p^0.
\]

Let \( J \in \mathcal{B}(\mathcal{Y}_p^0, \mathcal{Y}_p) \) be the "identification." Assume that for \( K = \mathcal{H}_0 J - JH \) we have the factorization \( K = \mathcal{X}_0 X, X \in C(\mathcal{Y}_p^0), X_0 \in C(\mathcal{Y}_p^0, \mathcal{Y}_p) \), in the sense that for \( f \in \mathcal{D}(\mathcal{H}_0) \cap \mathcal{D}(X), g \in \mathcal{D}(\mathcal{H}_0) \cap \mathcal{D}(X) \) we have

\[
(\mathcal{J}_g \mathcal{H}_0) - (J\mathcal{J}_g f, Jf) = (X_g, X_0 f).
\]

**THEOREM 1.** Let \( X_0 \mathcal{R}_0(z) \in \mathcal{B}(\mathcal{Y}_p^0, \mathcal{Y}_p) \), let \( X \in C(\Lambda_2(H; \Delta)) \), and assume that (1) holds. Then there exist the w.o., i.e. the strong limits

\[
(W_\pm(\mathcal{H}_0, H; J \mathcal{E}(\Delta))) = \lim_{t \to \pm \infty} \exp(it \mathcal{H}_0) J \mathcal{E}(\Delta) \exp(-it H).
\]

**THEOREM 2.** Let \( F = J*J - I \) and either \( F \in \Lambda_1(H; \Delta) \) or on a set of elements \( g \) dense in \( \mathcal{Y}_p^0 \) we have

\[
\lim_{\varepsilon \to 0} \int_\Delta \| FR(\lambda + i \varepsilon) g \|^2 d\lambda = 0.
\]

Then, under the assumptions of Theorem 1, the w.o. (2) are isometric on \( E(\Lambda) \).

We discuss now the invariance principle (see [6, 7]). Assume that the function \( \psi: \mathbb{R} \to \mathbb{R} \) is measurable and a.e. is finite and admissible on \( \Delta \), i.e. \( \psi \in C^1(\Delta); \psi'(\lambda) > 0, \lambda \in \Delta \); the variation of \( \psi' \) is bounded on \( \Delta \).

**THEOREM 3.** Under the assumptions of Theorem 1, for an admissible \( \psi \) there exist the w.o.

\[
(W_\pm(\mathcal{Y}_p^0, \mathcal{Y}_p^0; J \mathcal{E}(\Delta))) = W_\pm(\mathcal{Y}_p^0, H; J \mathcal{E}(\Delta)).
\]

3. Second-Order Equation with Respect to Time. The scattering for such equations is investigated with the aid of Theorem 3. Certain difficulties arise (see [8]) at the preservation of the previous identification. They can be overcome. Let \( H, \mathcal{H}_0 \) be strictly positive. Let \( J \) be the completion of \( \mathcal{D}(H) \) with respect to the norm \( \| H^{1/2} u \| \) and let

\[
\mathcal{Y}_p = \mathcal{Y}_p^{1/2} + \mathcal{Y}_p
\]

We consider in \( \mathcal{Y}_p \) the equation \( \dot{u} + Hu = 0 \). The resolving operator \( u(t) \in \mathcal{B}(\mathcal{Y}_p) \) associates to the pair \( \{u(0), \dot{u}(0)\} \) the pair \( \{u(t), \dot{u}(t)\} \). The operators \( u(t) \) form in \( \mathcal{Y}_p \) the unitary group. Similarly, for \( \mathcal{H}_0 \) one introduces \( \mathcal{Y}_0, u_0(t) \). Let \( E \) be the spectral measure for \( u(t); \Delta = [a, b], a > 0; \Delta = \{y \in \mathbb{R}: y \in \Delta\} \).

**THEOREM 4.** Let \( J \in \mathcal{B}(\mathcal{Y}_p^0, \mathcal{Y}_p^0) \), \( J = \mathcal{J} \mathcal{J} \). Under the assumptions of Theorem 1, there exist the w.o.

\[
(W_\pm(\mathcal{Y}_p^0, \mathcal{Y}_p^0; J \mathcal{E}(\Delta))) = W_\pm(\mathcal{Y}_p^0, \mathcal{Y}_p^0; J \mathcal{E}(\Delta)).
\]

Under the assumptions of Theorem 2, the w.o. (3) are isometric on \( E(\Delta) \).

Theorems 1-4 can be generalized in a straightforward manner to the case when instead of (1) one has a representation of the form

\[
K = \sum_i X_i^2 \chi_i \mathcal{X}_i.
\]

4. Requirements on the Domain. Below, \( \mathcal{O} \subseteq \mathbb{R}^n \), \( n \geq 2 \), \( \mathcal{O} = L^2(\mathcal{O}) \). The domain \( \mathcal{O} \) belongs to the class \( \mathcal{C}^2 \) and for some \( R_0 \) for \( \chi \in \partial \mathcal{O} \cap \{ |x| > R_0 \} \) one has \( \chi(0) = 0 \); here \( v(x) \) is the unit vector of the exterior normal. If \( \mathcal{O} \in \mathcal{O} \), then \( \sum \{ v(0) \} = \{ v \in S^1: v = \tau \omega \in \mathcal{O} \} \) does not decrease for \( r > R_0 \). We set \( \sum \{ v \} = \sum \{ v \} - \sum \{ v \} \) and we introduce the cone \( \mathcal{V}_a = \{ v \in \mathcal{R}^n: |x| \tau \in \mathcal{E}_a \} \).

\[
\text{dist}(x, \partial \mathcal{V}_a) = 0 (|x| \tau), \quad x \in \partial \mathcal{O}, \quad \tau < 1.
\]

The subsequent restrictions must ensure the presence of "cuts" with special properties and also the validity of certain estimates of the type of the imbedding theorems. We formulate these requirements directly. Let \( \chi_0 \) be the indicator of \( \mathcal{O} \); \( \zeta \in C^2(\mathcal{O}) \), \( 0 \leq \zeta(x) \leq 1 \), \( \zeta(x) = 0 \) near \( \partial \mathcal{O} \); \( \zeta(x) = \text{supp}(\chi_0 - \zeta) \). By \( I^\alpha \) we denote the weighted \( L^2 \)-norm in \( \mathcal{O} \) for the weight \( (1 + |x|)^\alpha \).

Condition By. For \( \mathcal{O} \in \mathcal{O} \) (4) holds. In \( \mathcal{O} \) there exists a truncating function \( \zeta \) such that