A Controllable Linear Stochastic System
With Delay in the Information Feedback Channel

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We consider the problem of optimal control of the solution of a linear stochastic differential equation whose stochastic terms depend on the solution and a control in the presence of variable delay in the information feedback channel. We find an explicit form for the optimal control minimizing a quadratic cost functional.

Bibliography: 6 titles.

In the present paper we study a controllable linear stochastic differential equation whose stochastic terms depend on the solution and the control. With complete observations the questions of optimal control of similar systems were studied by Wonham [1], Gihman and Skorohod [2], and Smirnov [3].

We consider the problem of optimal control of the solution of such an equation in the presence of variable delay in the information feedback channel. The method used to do this differs from the methods in the works just enumerated and is an extension of the methods of Bagchi and Kwakernaak [4] to more general controllable systems.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \((\mathcal{F}_t)_{t \geq 0}\) an increasing right-continuous family of \(\mathbb{P}\)-complete \(\sigma\)-algebras \(\mathcal{F}_t \subset \mathcal{F}\); \(w_r(t)\) and \(\nu_r(t, C)\), \(t \in [0, T]\), \(C \in \mathcal{B}(\mathbb{R}^{m_r}) \setminus \{0\}\), \(r = 1, 2, 3\), and jointly independent standard \(n_r\)-dimensional Wiener processes and Poisson measures with respect to the family \((\mathcal{F}_t)\); \(\nu_r(\cdot)\) is a \(\sigma\)-finite measure in the representation \(M\nu_r(t, C) = t\nu_r(t, C), \nu_r(t, C) = \nu_r(t, C) = t\nu_r(t, C)\) (cf. [2]).

In what follows we shall use the following notation: \(\mathcal{D}\) is the space of functions \(g : [0, T] \to \mathbb{R}^n\) that are continuous from the right and have limits from the left, \(\mathcal{L}_d\) is the space of square \((d \times d)\) matrices, \(\hat{\mathcal{L}}_d\) is the space of measurable integrable functions assuming values in \(\mathcal{L}_d\), \(\mathbb{E}_d\) is the unit matrix in \(\mathcal{L}_d\), and * indicates transposition. If \(A\) and \(A'\) belong to \(\mathcal{L}_d\), then the inequality \(A \geq A'\) means that \(A - A'\) is nonnegative definite. If \(A \in \hat{\mathcal{L}}_d\), then \(\Phi_A(t, s)\) is a solution of the equation
\[
\frac{d}{dt} \Phi_A(t, s) = A(t) \Phi_A(t, s), \quad \Phi(s, s) = \mathbb{E}_d.
\]

Consider the controlled stochastic differential equation
\[
\xi(t) = \xi_0 + \int_0^t (A(s)\xi(s) + B(s)\eta(s)) \, ds + \mu_t(\xi, \eta); \quad t \in [0, T],
\]
where
\[
\mu_t(\xi, \eta) = \int_0^t G_1(s, \xi(s)) \, dw_1(s) + \int_0^t \int g_1(s, \xi(s), y) \, \nu_1(ds, dy) + \int_0^t G_2(s, \eta(s)) \, dw_2(s)
+ \int_0^t \int g_2(s, \eta(s), y) \, \nu_2(ds, dy) + \int_0^t G_3(s) \, dw_3(s) + \int_0^t \int g_3(s, y) \, \nu_3(ds, dy).
\]
and the quadratic control cost functional is
\[
\mathcal{J}(\eta) = M\{ \int_0^T \xi^*(t)H(t)\xi(t) + \eta^*(t)N(t)\eta(t) \, dt + \xi^*(T)H_T\xi(T) \}.
\]

Here $\xi(t)$ assumes values in $\mathbb{R}^n$, $\eta(t)$ assumes values in $\mathbb{R}^m$, $\xi_0$ is $\mathcal{F}_0$-measurable, and $M|\xi_0|^2 < \infty$.

We assume that

\[
G_1(t, x) = \sum_{i=1}^{n} x_i G_{1,i}(t); \quad g_1(t, x, y) = \sum_{i=1}^{n} x_i g_{1,i}(t, y);
\]

\[
G_2(t, u) = \sum_{j=1}^{m} u_j G_{2,j}(t); \quad g_2(t, u, y) = \sum_{j=1}^{m} u_j g_{2,j}(t, y),
\]

where $x_i$ and $u_j$ are the components of the vectors $x$ and $u$. The coefficients $A(t)$, $B(t)$, $G_{1,i}(t)$, $i = 1, n$, $G_{2,j}(t)$, $j = 1, m$, $G_3(t)$, $H(t)$, and $N(t)$ are continuous matrices of the corresponding dimensions; $g_{1,i}(t, y)$, $i = 1, n$, $g_{2,j}(t, y)$, $j = 1, m$, and $g_3(t, y)$ are measurable $n$-dimensional vector-valued functions such that

\[
\int (g_{r,i} G^*_{r,i})(t, y) q_r(dy), \quad r = 1, 2, \quad \text{and} \quad \int (g_3 G^*_3)(t, y) q_3(dy)
\]

are continuous. The symmetric matrices $H(t)$, $N(t)$, and $H_T$ satisfy the conditions $H(t) \geq 0$, $H_T \geq 0$, $N(t) \geq \varepsilon E_m$, with $\varepsilon > 0$.

Let $h(t)$ be a monotonically nondecreasing continuous function with $0 \leq h(t) \leq t$. We assume that at each instant of time $s$ one observes $\xi(h(s))$ and the control $\eta$ at the instant $t$ can use the observations on the time interval $[0, t]$. The quantity $t - h(t)$ is the delay at the instant $t$ in the information feedback channel.

We denote by $X$ the space of nonanticipating functionals $f(t, g) : [0, T] \times \mathcal{D} \to \mathbb{R}^m$ such that the equation

\[
\xi(t) = \xi_0 + \int_0^t (A(s)\xi(s) + B(s)f(s, \xi(h(s)))) ds + \mu(t, \xi, f(\cdot, \xi(h)) \tag{2}
\]

has a unique solution and $\int_0^T |f(t, \xi(h(s))))| dt < \infty$ almost surely. The class of admissible controls $U$ is defined as the set of processes $\eta(t) = f(t, \xi(h(s)))$, where $f \in X$ and $\xi$ is a solution of Eq. (2). We denote by $\tilde{U}$ the class of measurable and $(\mathcal{F}_{h(t)})$-adapted processes $\eta$ with $\int_0^T |\eta(t)|^2 dt < \infty$ almost surely.

We note that if $\eta \in \tilde{U}$ then there exists a unique solution $\xi$ of Eq. (1) with trajectories in $\mathcal{D}$ and $\mu_t(\xi, \eta)$ is a local square-integrable $(\mathcal{F}_t)$-martingale with characteristic

\[
(\mu(\xi, \eta))_t = \int_0^t ((G_1 G^*_1)(s, \xi(s)) + \int (g_1 G^*_1)(s, \xi(s), y) q_1(dy)) ds + \int_0^t ((G_2 G^*_2)(s, \eta(s)) + \int (g_2 G^*_2)(s, \eta(s), y) q_2(dy)) ds + \int_0^t ((G_3 G^*_3)(s) + \int (g_3 G^*_3)(s, y) q_3(dy)) ds.
\]

It is obvious that $U \subset \tilde{U}$. It will be shown below that the control that is optimal in $\tilde{U}$ belongs to $U$.

Assuming that $F \in \mathcal{L}_n$ we introduce the following notation:

\[
\Delta(t, s, F) = \text{tr} \{G^*_3(s) \Phi_A^*(t, s) F \Phi_A(t, s) G_3(s) + \int g_3^*(s, y) \Phi_A^*(t, s) F \Phi_A(t, s) g_3(s, y) q_3(dy)\};
\]

\[
\Delta(t, s, F) \quad \text{and} \quad \Gamma(t, s, F)
\]

are matrices of dimensions $n \times n$ and $m \times m$ whose $(i, j)$ elements are

\[
\text{tr} \{G^*_{r,i}(s) \Phi_A^*(t, s) F \Phi_A(t, s) G_{r,j}(s) + \int g_{r,i}^*(s, y) \Phi_A^*(t, s) F \Phi_A(t, s) g_{r,j}(s, y) q_r(dy)\}
\]

respectively for $r = 1$ and $r = 2$.

We note some properties of the matrices $\Delta$ and $\Gamma$.

1. $\Delta(t, s, \cdot)$ is a linear mapping of $\mathcal{L}_n$ into itself; $\Gamma(t, s, \cdot)$ is a linear mapping of $\mathcal{L}_n$ into $\mathcal{L}_m$.
2. If $F \geq 0$, then $\Gamma(t, s, F)$ $\geq 0$ and $\Delta(t, s, F) \geq 0$. 

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