ON THE SUMMABILITY OF FUNCTIONS ANALYTIC IN STAR-SHAPED DOMAINS

V. A. Belyaev

Let \( f(z) \) be a function which is analytic over the whole plane except for an arbitrary interval \( \Delta, \Delta \not\subset \Delta_n \). Suppose that \( f(z) \) has a power series expansion about the origin. We prove that a matrix exists which sums the power series to \( f(z) \) over the whole plane except for \( \Delta \).

We consider a function \( f(z) \), analytic in the domain \( M = \mathbb{C} \setminus \Delta, \Delta = U\Delta (r_\alpha, \varphi_\alpha), 1 \leq \alpha \leq \nu_0, \Delta (r_\alpha, \varphi_\alpha) = \{z = r \exp \ i\varphi, 0 < r < r_\alpha \leq \infty \}, \) which may be expanded about the origin in the power series

\[
\sum_{k=0}^{\infty} c_k z^k.
\]

It is known (see [1]) that a \( \gamma \)-matrix \( \{g_k (\alpha)\}_{\alpha=0}^{\infty} \) exists, which sums the series (1) to \( f(z) \) in the domain \( M \), i.e.,

\[
f(z) = \lim_{\omega \to \infty} \sum_{k=0}^{\infty} g_k (\omega) c_k z^k,
\]

where on each compactum in the interior of \( M \) the convergence is uniform.

P. L. Ul'yanov has posed the following problem: let the function \( f(z) \) be analytic in the domain \( \mathbb{C} \setminus \{1\} \). Does there exist a matrix which sums the power series (1) corresponding to \( f(z) \) to \( f(z) \) in the domain \( \mathbb{C} \setminus \{1\} \)? We find that the answer to this question is in the affirmative under very weak conditions imposed on \( f(z) \).

**Lemma 1.** If the function \( f(z) \) is defined over the whole plane \( \mathbb{C} \), is analytic in the domain \( \mathbb{C} \setminus \Delta = M \), and if for each point \( z = r \exp \ i\varphi \) of the set \( \Delta \setminus F, F \subset \Delta, \lim_{\alpha} f(t) = f(z), q_\alpha \leq \text{arg} \ t < \varphi_\alpha + \pi, \) then for an arbitrary \( \varepsilon > 0 \) and an arbitrary bounded domain \( B, B \subset \mathbb{C} \),

\[
D(R, \varepsilon) = \{ |z| \leq R \setminus \bigcup_{\alpha} \{z - (r_\alpha - \varepsilon \exp \ i\varphi_\alpha) \leq 0 \} (1 \leq \nu \leq \nu_0) \}
\]

we can find a number \( N \), such that for \( n > N \) the inequality \( |f(z) - f\left(z \exp \frac{i}{n}\right)| < \varepsilon \) holds for all \( z \in B \).

The proof of this lemma proceeds in the same way as the proof of Cantor's theorem concerning the uniform continuity of a function continuous on an interval.

**Theorem 1.** If the function \( f(z) \) is defined over the whole plane, is analytic in the domain \( M = \mathbb{C} \setminus \Delta (1, 0) \), and if for each point \( z \) of the set \( \Delta (1, 0) \setminus F, F \subset \Delta (1, 0), \lim_{z \to \infty} f(z) = f(z), 0 \leq \text{arg} \ z < \pi, \) then there exists a matrix \( \{g_k (\omega)\}_{\omega=0}^{\infty} \) which sums the series (1) corresponding to \( f(z) \) to \( f(z) \) at each point of the domain \( \mathbb{C} \setminus F \); moreover, in each bounded domain \( B, B \subset M \),


© 1975 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.
tends towards zero uniformly.

**Proof.** If we consider the function \( f(z \exp \frac{i}{n}) \), for which \( z = \exp \left( - \frac{i}{n} \right) \) is a singular point, then, in accordance with relation (2),

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} g_k(\omega) \left[ \exp \left( \frac{ik}{n} \right) \right] c_k z^k = f \left( z \exp \frac{i}{n} \right)
\]

over the whole plane except for the points \( \Delta \left( 1, - \frac{i}{n} \right) \). For each \( n \) we consider the disk \( \delta_n = \left\{ \left| z - \exp \left( - \frac{i}{n} \right) \right| < r_n \} \), where \( r_n \) is such that \( 1 \in \delta_n \). It follows from this that \( r_n \to 0 \) for \( n \to \infty \). We draw two rays \( \Gamma_1^{(n)} \) and \( \Gamma_2^{(n)} \), tangent to the circle \( \left| z - \exp \left( - \frac{i}{n} \right) \right| = r_n \), parallel to and having the same direction as \( \Delta \left( 1, - \frac{i}{n} \right) \); the origin of the rays is at their points of tangency. We consider the set \( F_n = \delta_n \cup G_n \), where \( G_n \) is the domain bounded by \( \left| z - \exp \left( - \frac{i}{n} \right) \right| = r_n \), and by the rays \( \Gamma_1^{(n)} \) and \( \Gamma_2^{(n)} \), \( 0 \in G_n \). Let \( D_n = CF_n \cap \{ |z| < n \} \), and let \( CF_n \) be the complement of \( F_n \) with respect to the whole plane. For \( D_n \) there exists a \( \omega_n \), such that for \( \omega \geq \omega_n \) the inequality

\[
\left| \sum_{k=0}^{\infty} g_k(\omega) \left[ \exp \left( \frac{ik}{n} \right) \right] c_k z^k - f \left( z \exp \frac{i}{n} \right) \right| < \frac{1}{n}.
\]

is satisfied. There is no loss of generality if we assume that \( \omega_n > n \) and \( \omega_1 < \omega_2 < \omega_3 < \ldots \). Let us put

\[
g_k(\omega) = g_k(\omega) \exp \left( \frac{ik}{n} \right), \quad \omega_0 < \omega < \omega_{n+1} \quad (n = 1, 2, 3, \ldots).
\]

We show that the matrix (4) is the desired matrix. We show first that the matrix (4) sums the series (1), in an arbitrary domain interior to \( M \), uniformly to \( f(z) \). We consider an arbitrary domain \( \overline{B}, B \subset M \), i.e., \( B \cap F = \phi \). We can take \( \varepsilon \) and \( R \) such that \( \overline{B} \subset D(R, \varepsilon) \). According to Lemma 1, for arbitrary preassigned \( \varepsilon > 0, R > 0 \), there exists an \( N_1 \), such that for \( n > N_1 \) and for all \( z \in B \), we have the inequality

\[
\left| f(z) - f \left( z \exp \frac{i}{n} \right) \right| < \varepsilon.
\]

Further, there exists an \( N_2 \), such that \( B \cap \left\{ \bigcup_{n=N_2}^{\infty} F_n \right\} = \phi \) and \( B \subset D_{N_2} \). Consequently, for \( n > N_2 \), \( \omega_n \leq \omega < \omega_{n+1} \), we have

\[
\left| \sum_{k=0}^{\infty} g_k(\omega) c_k z^k - f \left( z \exp \frac{i}{n} \right) \right| < \frac{1}{n}, \quad z \in B.
\]

Taking into account the inequalities (5) and (6), we obtain

\[
\left| f(z) - \sum_{k=0}^{\infty} g_k(\omega) c_k z^k \right| \leq \left| f(z) - f \left( z \exp \frac{i}{n} \right) \right| + \left| f \left( z \exp \frac{i}{n} \right) - \sum_{k=0}^{\infty} g_k(\omega) c_k z^k \right| < \varepsilon + \frac{1}{n},
\]

\[
n > N_1 + N_2, \quad \omega_0 \leq \omega < \omega_{n+1}, \quad z \in B.
\]

The second assertion of the theorem follows from this. It remains to show that the matrix (4) sums the series (1) to \( f(z) \) at each point of the set \( \Delta \left( 1, 0 \right) \setminus F \). Let \( f(z) \in \Delta \left( 1, 0 \right) \setminus F \). A domain \( \overline{B}_1 \) exists such that \( z_0 \in B_1 \) and \( \overline{B}_1 \) satisfies the conditions of Lemma 1. Analogous to the previous proof we may prove the existence of a number \( N_3 \), such that for \( n > N_3 \) and all \( z \in \overline{B}_1 \), the inequalities (5) and (6) are satisfied simultaneously. Consequently, these inequalities are satisfied for \( n > N_3 \) even at the point \( z = z_0 \), whence the first assertion of the theorem follows. Thus the proof of the theorem is complete.