SOME GENERALIZATIONS OF THE RIEMANN SPACES OF EINSTEIN

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We introduce a class of Riemann structures, called generalized Einstein structures of index 2e, of which Einstein spaces are a particular case. We show that these structures are stationary for functions introduced on a family of Riemann structures of the compact manifold of H. Weyl. This result solves a problem of M. Berger. As examples of structures which are generalized Einstein structures over all indices we cite homogeneous compact Riemann spaces with a nondecomposable isotropy group and products of such spaces.

In this paper we consider a number of functions of Riemann structures on a compact manifold, including the integral of the scalar curvature (k2). We introduce a new class of structures, called generalized Einstein structures of index 2e, which are stationary for the functions considered, much as the Einstein structures are stationary for the function k2. By the same token, we solve one of the problems posed by M. Berger [1]. We prove that structures which are generalized Einstein structures over all indices are invariant structures on homogeneous compact spaces with a nondecomposable isotropy group and on products of such spaces.

Let M be a compact manifold of dimension n; let $\mathcal{R}$ be a set of Riemann structures on M; and let $\mathcal{R}_1$ be the set of Riemann structures with unit volume.

Let $g \in \mathcal{R}_1$ and let $R = R_{ij}^{kl}$ be the curvature tensor of the structure $g$. We consider the following functions of the curvature tensor [2]:

$$H_{2e} = \sum_{i}^{n} h_{i}^{k} h_{i}^{l} R_{k}^{i} \cdot \ldots \cdot R_{l}^{i} h_{i},$$

$$k_{2e} = \int_{M} H_{2e} \cdot \nu.$$  

Here $\delta^{i}_{i}$ is the Kronecker symbol, and $\nu$ is the volume element of the structure $g$; the tensor indices are raised and lowered in the usual way with the help of the metric tensor.

It is a known fact that if $n = 2e$, then $k_{2e} = k_2$ is a topological invariant of the manifold M (to within a multiplier this is the Euler characteristic); for $n < 2e$, $k_{2e} = 0$, but for $n > 2e$, $k_{2e}$ depends on the structure and, consequently, is a function on $\mathcal{R}_1$.

Following Berger [1], we enclose our structure in an arbitrary one-parameter family of structures $g(t) \in \mathcal{R}_1$, $g(0) = g$, and we find a necessary condition for an extremum of the function $k_{2e}(g(t)) = k_{2e}(t)$.

We introduce the notation $(dg_{ij})/dt = h_{ij}$. Then the following formulas hold (see [1]; a prime indicates differentiation with respect to $t$):

$$(\Gamma^{i}_{ij})' = Z_{ij}^{-} = \frac{1}{2} (\nabla h_{i}^{k} + \nabla h_{j}^{k} - \nabla h_{k}^{i},)$$
We find it convenient to rewrite the function $H_{2e}$ in the form

$$H_{2e} = R_{i_b j_b}^{i_b h} \ldots R_{i_e j_e}^{i_e h}.$$  

The square brackets denote alternation over all the lower indices. Summation is carried out with respect to identical indices. We introduce the new tensor

$$T_{2e} = \langle T_{2e} \rangle = \langle T_{2e} \rangle.$$  

**THEOREM 1.**

$$\langle \rangle$$ denotes the global scalar product of tensors (the integral of their complete convolution).

**Proof.** To simplify the derivation we consider only orthonormal frames of reference and we arrange the tensor indices as may be convenient, above or below. We remark also that we can always replace an alternation on the lower indices by an alternation on the same indices appearing above.

Applying the rule for differentiating a product and Eqs. (1), we obtain:

$$H_{2e} = e \cdot R_{i_b j_b}^{i_b h} \cdot R_{i_b j_b}^{i_b h} \ldots - e \langle h, T_{2e} \rangle = -e \langle h, T_{2e} \rangle.$$  

$\langle \rangle$ denotes the convolution of two tensors (local scalar product),

$$A = e \langle h, T_{2e} \rangle.$$  

The first term of this sum is equal to zero since the alternation takes place with respect to the indices $i_b, j_b$ of the symmetric tensor $h$; the second and third terms are equal to each other.

We now go to the function $k_{2e}$:

$$k_{2e} = \left( \sum_M H_{2e} \cdot v \right) = \sum_M H_{2e} \cdot v + \sum_M H_{2e} \cdot v = 2e \sum_M \nabla R_{i_b j_b}^{i_b h} \cdot R_{i_b j_b}^{i_b h} \ldots + e \langle h, T_{2e} \rangle + \frac{1}{2} H_{2e} \langle g, h \rangle.$$  

By Stokes' Theorem, the first term is equal to

$$-2e \sum_M \nabla R_{i_b j_b}^{i_b h} \cdot \nabla (R_{i_b j_b}^{i_b h} \ldots) \cdot v = 0,$$

since alternation over all the lower indices includes Bianchi's identity:

$$\nabla R_{i_b j_b}^{i_b h} + \nabla R_{i_b j_b}^{i_b h} + \nabla R_{i_b j_b}^{i_b h} = 0.$$  

This completes the proof of the theorem.

**COROLLARY.** In order for the function $k_{2e}$ to have an extremum at a point $g \in \mathbb{R}$, it is necessary that

$$T_{2e} = \frac{H_{2e}}{n} \cdot g.$$  

**Proof.** We have here a problem concerning a conditional extremum: the condition is the requirement of unit volume. Solving this problem by Lagrange's method, we obtain