VIBRATIONS OF A SEMICIRCULAR MEMBRANE CONTAINING THIN RIGID INCLUSIONS

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This article examines problems concerning steady-state vibrations of a semicircular membrane containing thin rigid inclusions of different configurations. The generalized method of integral transforms is used to formulate the problem in the form of a system of singular integral equations in each specific case. With the use of the asymptote of the sought functions as a basis, these equations are solved approximately by the method of orthogonal polynomials. A study is made of the validity of using the reduction method to approximately solve the infinite linear algebraic matrix system which is obtained. The results of calculations are analyzed.

We will examine the vibrations of a membrane containing rigid inclusions of different configurations. In each case, we use the generalized method of integral transforms to reduce the problem to a system of singular integral equations which is solved approximately on the basis of the asymptote. The solutions are obtained by the method of orthogonal polynomials. A study is made of the validity of using the reduction method to approximately solve the infinite linear algebraic matrix systems that are obtained. The results of calculations are analyzed.

1. We will examine the steady-state vibrations of a semicircular membrane \((0 \leq r \leq R, \ 0 \leq \theta \leq \pi)\) fastened at its contour and containing two rigid inclusions located along the lines of a polar coordinate grid; the inclusions are arcuate \((r = c, a_1 \leq \theta \leq b_1)\) and radial \((\theta = \gamma, a_2 \leq r \leq b_2)\). The inclusions are subjected to loads which are periodic in time \(P_1(\theta, t) = P_1(\theta)\cos \omega t, \ P_2(r, t) = P_2(r)\cos \omega t\). Since the forced vibrations take place with the frequency of the exciting force, the mathematical problem is formulated relative to the function of the amplitudes of the deflections of the membrane \(w(r, \theta)\)

\[
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \mu^2 w = 0 \quad (0 \leq r \leq R, \ 0 \leq \theta \leq \pi);
\]

\[
w (R, \theta) = 0 \quad (0 \leq \theta \leq \pi);
\]

\[
w (r, \pi) = w (r, 0) = 0 \quad (0 \leq r \leq R);
\]

\[
w (c, \theta) = f_1 (\theta) = A_1 \cos \theta + B_1 \sin \theta + D_1 \quad (a_1 \leq \theta \leq b_1);
\]

\[
w (r, \gamma) = f_2 (r) = A_2 r + B_2 \quad (a_2 \leq r \leq b_2).
\]

Here, \(\mu^2 = T \omega^2 \rho_0^{-1}\); \(T\) is the tension of the membrane; \(\rho_0\) is the surface density of the membrane; \(\omega\) is the frequency of the vibrations; \(A_j, B_j\) \((j = 1, 2)\), \(D_1\) are constants to be determined.

The deflections of the membrane are continuous in the transition through the inclusion, while the normal derivatives undergo unknown discontinuities

\[
\left(\frac{\partial w}{\partial r}\right) (c, \theta) = \frac{\partial w}{\partial r} (c - 0, \theta) - \frac{\partial w}{\partial r} (c + 0, \theta) = \begin{cases} \chi_1 (\theta), & 0 \in [a_1, b_1]; \\ 0, & \theta \notin [a_1, b_1]; \end{cases}
\]

\[
\frac{1}{r} \left(\frac{\partial w}{\partial \theta}\right) (r, \gamma) = \frac{1}{r} \frac{\partial w}{\partial \theta} (r, \gamma - 0) - \frac{1}{r} \frac{\partial w}{\partial \theta} (r, \gamma + 0) = \begin{cases} \chi_2 (r), & r \in [a_2, b_2]; \\ 0, & r \notin [a_2, b_2]. \end{cases}
\]
Thus, it is necessary to find the unknown functions $w(r, \theta)$, $x_1(\theta)$, $x_2(r)$ and the constants $A_j$, $B_j$ ($j = 1, 2$, $D_1$.

2. We will apply the finite Fourier cosine transform to Eqs. (1)-(3) with respect to the variable $\theta$. Using the generalized method of integral transforms [1] and taking (6)-(7) into account, we arrive at a discontinuous boundary-value problem relative to the transform

$$w''(r) + \frac{1}{r}w'(r) + \left(\mu^2 - \frac{n}{r^2}\right)w(r) = -\frac{1}{r^2} \frac{\partial w}{\partial \theta}(r, \gamma) \sin n\gamma;$$

$$w_n(0) = w_n(R) = 0;$$

$$\langle w_n(\omega) \rangle = \chi_{1n} = \int_0^{b_1} \chi_1(\eta) \sin n\eta d\eta.$$

We represent discontinuous solution (8)-(10) in the form

$$w_n(r) = -\int_0^R G_n(r, \xi) e^{-\int_0^\xi \left(\frac{\partial R}{\partial \theta}(\xi, \gamma)\right) \sin n\gamma d\xi} - \chi_{1n} G_n(r, \xi);$$

$$G_n(r, \xi) = -\frac{\pi\xi J_n(\xi)}{2J_n(\xi)} + \frac{\pi\xi N_n(\xi)}{2N_n(\xi)} + \frac{\pi\xi}{2} \left\{ J_n(\xi) N_n(\xi), (r > \xi); \right.$$  

$$\left. J_n(\xi) N_n(\xi), (r < \xi), \right.$$  

where $G_n(r, \xi)$ is the Green’s function of problem (8)-(9) constructed in accordance with the cited work. Obtaining the original from the formula for inversion of the finite cosine transform, taking (10) into account, and changing the order of summation and integration, we obtain

$$w(r, \theta) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \left[ 2 \sum_{n=1}^\infty G_n(r, \xi) \sin n\theta \sin n\eta \right] \chi_1(\eta) d\eta =$$

$$-\frac{1}{\pi} \sum_{n=1}^{\infty} \left[ 2 \sum_{n=1}^\infty G_n(r, \xi) \sin n\theta \sin n\eta \right]$$

Using Gräffe’s theorem for cylindrical functions [2] and the well-known properties of a Neumann function, we can show that

$$\sum_{n=1}^\infty G_n(r, \xi) \cos n(\gamma - \theta) = \frac{\xi}{2} \ln R_0(r, \theta) + \frac{\xi}{2} Q(r, \xi, \gamma - \theta),$$

where $R_0(r, \theta) = (r^2 + c^2 - 2rc \cos(\gamma - \theta))^{1/2}$, $Q(r, \xi, \gamma - \theta)$ is a continuous function whose first derivatives with respect to $r$ and $\xi$ have a finite discontinuity at $r = \xi$.

We transform (11) with allowance for (12), then realizing conditions (4) and (5). As a result, we obtain a system of integral equations relative to the functions $\chi_1(\theta)$, $\chi_2(r)$

$$\sum_{k=1}^{J} \left[ \delta_{jk} \ln|x - \xi| + L_{jk}(x, \xi) \right] \chi_k(\xi) d\xi = -4\pi j \theta,$$

where $L_{jk}(x, \xi)$ are continuous functions having finite first derivatives; to save space, we will not present the expressions for these functions here.

Having made a substitution of variables which leads us to the interval $[0, 1]$ and maps the points $x = b$ ($j = 1, 2$) on the point $t = 0$, we differentiate both equations with respect to the external variable. This gives us

$$\sum_{k=1}^{2} \int_0^1 \left| \frac{\partial^2}{\partial \tau^2} \sin \frac{\lambda_k}{\tau} \right| \chi_k'(\tau) d\tau = -2\pi \rho j \theta;$$

where $\rho = \frac{a_2 - a_1}{b_2 - b_1}$. 

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