ORTHOGONAL BASES FOR LP SPACES

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The spectrum of a system of functions which are orthogonal on \([0, 1]\) is the set of all \(p \in [1, \infty]\) such that the system forms a basis in \(L^p[0, 1]\) \((L^\infty = C)\). A set \(E\) is called a spectral set if there exists a system of functions orthonormal on \([0, 1]\) whose spectrum is \(E\). In this note we determine all spectral sets and construct an orthonormal system corresponding to each of them.

A. N. Slepchenko [1] showed that there exists a system of functions which is orthonormal on \([0, 1]\) and forms a basis for the space \(L[0, 1]\) but which is not closed in \(L^2[0, 1]\). We will call the set of all \(p \in [1, \infty]\) such that a system of functions orthogonal on \([0, 1]\) forms a basis for the space \(L^p[0, 1]\) \((L^\infty = C)\) the spectrum of that system. We will call a set \(E\) spectral if there exists a system of functions orthonormal on \([0, 1]\) whose spectrum is \(E\). In these terms the result in [1] means that there exists a spectral set \(E\) such that \(1 \in E\) and \(2 \notin E\).*

In the present note all spectral sets \(E\) will be determined and for each of them a corresponding orthonormal set will be constructed.

§1. NOTATION AND LEMMAS

Fix a set \(P\), either \([1, p)\) \((1 < p \leq 2)\) or \([1, p]\) \((1 \leq p < 2)\), and sequences of positive numbers \(a = \{a_k\}\), \(b = \{b_k\}\) such that

\[
\sum_{k=1}^{\infty} b_k = 1, \tag{1}
\]

\[
|a_k b_k|_p = \begin{cases} \infty, & \text{if } a \in P', \\ < \infty, & \text{if } a \notin P', \end{cases}
\]

where \(P'\) denotes the set of numbers which are conjugate to the numbers in \(P\) in the sense of Young \(P(\|x\|_P = \|x\|_{P'}, l_\infty = m)\). For example, if \(P = [1, p)\) \((1 < p \leq 2)\), then we may take

\[
a_k = 2^{\frac{k}{p} - \frac{k}{p'}}, \quad k = 1, 2, \ldots, \left(\frac{1}{p} + \frac{1}{p'} = 1\right), \quad b_k = 2^{-\frac{k}{p}}, \quad k = 2, 3, \ldots, \quad b_1 = 1 - \sum_{k=2}^{\infty} 2^{-\frac{k}{p}} \tag{3}
\]

or

\[
a_k = 2^{\frac{k}{p} - \frac{k}{p'}}, \quad b_k = 2^{-k}. \tag{4}
\]

If \(P = [1, p]\) \((1 \leq p < 2)\), then we may set

\[
a_k = 2^{\frac{k}{p} + \frac{k}{p'}}, \quad k = 1, 2, \ldots,
\]

\[
b_k = 2^{-\frac{k}{p'}}, \quad k = 2, 3, \ldots, \quad b_1 = 1 - \sum_{k=2}^{\infty} 2^{-\frac{k}{p'}} \tag{5}
\]

*After this note had been prepared in manuscript form for publication, an article by B. V. Ryazanov and A. N. Slepchenko [5] appeared in which they determined all spectral sets on \([1, 2]\). However, their proof does not enable one to construct orthonormal systems corresponding to the given spectrum.

or
\[ a_k = 2^{3k}, \quad b_k = 2^{-k}. \] (6)

Fix a system of nonintersecting intervals \( E_k \subseteq [0, 1] \) such that \( |E_k| = b_k \), and set
\[ \phi_n(t, a, b) = \phi_n(t) = \begin{cases} 0, & \text{if } t \in E_k, 1 \leq k < n, \\ a_k b_k^{-1}, & \text{if } t \in E_n, \\ -R_\alpha a_k, & \text{if } t \in E_k, k > n, \end{cases} \] (7)
where \( R_n = \sum_{k=1}^{n} a_k^2 b_k \) (see [2]), and
\[ \phi^*_n(t, a, b) = \phi^*_n(t) = \begin{cases} R_{n-1} a_k^2 b_n a_n, & \text{if } t \in E_k, 1 \leq k < n, \\ a_n, & \text{if } t \in E_n, \\ 0, & \text{if } t \in E_k, k > n. \end{cases} \] (8)

We note (see [2]) that
\[ \phi_n \in L^\alpha \quad \text{for } x \notin P', \quad \phi_n \notin L^\alpha \quad \text{for } x \in P' \quad (L^\alpha = M), \quad \phi_n \in M. \] (9)

Let \( \chi^{(k)}(i = 1, 2, \ldots) \) be the functions of the Haar system constructed on the \( E_k \) and extended by zero to \([0, 1]\). Set
\[ \psi_0(t, a, b) = \psi_0(t) = -R_0 a_n, \quad \psi_n(t, a, b) = \psi_n(t) = \begin{cases} \phi_n(t), & \text{if } t \in E_k \quad n = 1, 2, \ldots; \\ \chi^{(n+1)}(t) = \begin{cases} \phi^*_n(t), & \text{if } t \in E_k, 1 \leq k < n, \\ \chi^{(k+1)}(t), & \text{if } t \in E_k, k > n, \end{cases} \end{cases} \] (10)

We note [see (2)] that
\[ \text{Lemma 1. The system } \{\psi_n\}_{n=0}^{\infty} \text{ is orthogonal on } [0, 1] \text{ and closed in } L^\alpha[0, 1] \text{ for } \alpha \notin P'. \]

Proof. Note that
\[ \int_0^1 \phi_n(t) \chi^{(n)}(t) dt = \int_{E_k} \phi_n(t) \chi^{(n)}(t) dt = \langle \phi_n, \chi^{(n)} \rangle \int_{E_k} \chi^{(n)}(t) dt = 0. \]
\[ n = 1, 2, \ldots; k = 1, 2, \ldots; \quad \text{if } (k, l) \neq (k_1, l_1). \]

Furthermore,
\[ \int_0^1 \chi^{(n)}(t) \chi^{(n)}(t) dt = \int_{E_k \cap E_k} \chi^{(n)}(t) \chi^{(n)}(t) dt = 0 \]
if \( (k, l) \neq (k_1, l_1) \). Therefore, in order to prove orthogonality it suffices to verify that \( \langle \phi_m, \phi_n \rangle = 0 \) for \( m = n, m = 0, 1, \ldots, n = 0, 1, \ldots. \) Since for \( m < n \)
\[ \phi_m(t) \phi_n(t) = \begin{cases} 0, & \text{if } t \in E_k, 1 \leq k < n, \\ -R_m a_k^2 b_n, & \text{if } t \in E_n, \\ R_m a_k^2 b_n, & \text{if } t \in E_k, k > n, \end{cases} \]