ORTHOGONAL BASES FOR LP SPACES

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The spectrum of a system of functions which are orthogonal on [0, 1] is the set of all \( p \in [1, \infty] \) such that the system forms a basis in \( L^p[0, 1] \) (\( L^\infty = C \)). A set \( E \) is called a spectral set if there exists a system of functions orthonormal on [0, 1] whose spectrum is \( E \).

In this note we determine all spectral sets and construct an orthonormal system corresponding to each of them.

A. N. Slepchenko [1] showed that there exists a system of functions which is orthonormal on [0, 1] and forms a basis for the space \( L[0, 1] \) but which is not closed in \( L^2[0, 1] \). We will call the set of all \( p \in [1, \infty] \) such that a system of functions orthogonal on [0, 1] forms a basis for the space \( L^p[0, 1] \) (\( L^\infty = C \)) the spectrum of that system. We will call a set \( E \) spectral if there exists a system of functions orthonormal on [0, 1] whose spectrum is \( E \). In these terms the result in [1] means that there exists a spectral set \( E \) such that \( 1 \notin E \) and \( 2 \notin E \).*

In the present note all spectral sets \( E \) will be determined and for each of them a corresponding orthonormal set will be constructed.

§1. NOTATION AND LEMMAS

Fix a set \( P \), either \( [1, p) \) (\( 1 < p \leq 2 \)) or \( [1, p] \) (\( 1 \leq p < 2 \)), and sequences of positive numbers \( a = \{a_k\} \), \( b = \{b_k\} \) such that

\[
\sum_{k=1}^{\infty} b_k = 1,
\]

(1)

\[
|a_k b_k^{1/k}| = \begin{cases} \infty, & \text{if } \alpha \in P', \\ < \infty, & \text{if } \alpha \notin P', \end{cases}
\]

(2)

where \( P' \) denotes the set of numbers which are conjugate to the numbers in \( P \) in the sense of Young \( P(\|x\|_\alpha = \|x\|_\infty, \theta, \omega = m) \). For example, if \( P = [1, p) \) (\( 1 < p \leq 2 \)), then we may take

\[
a_k = 2^{k^p/k}, \quad k = 1, 2, \ldots \left( \frac{1}{p} + \frac{1}{p'} = 1 \right), \quad b_k = 2^{-k'}, \quad k = 2, 3, \ldots, b_1 = 1 - \sum_{k=2}^{\infty} 2^{-k'}
\]

(3)

or

\[
a_k = 2^{k^p/k}, \quad b_k = 2^{-k}.
\]

(4)

If \( P = [1, p] \) (\( 1 \leq p < 2 \)), then we may set

\[
a_k = 2^{k^p/k}, \quad k = 1, 2, \ldots,
\]

\[
b_k = 2^{-k}, \quad k = 2, 3, \ldots, b_1 = 1 - \sum_{k=2}^{\infty} 2^{-k'}
\]

(5)

*After this note had been prepared in manuscript form for publication, an article by B. V. Ryazanov and A. N. Slepchenko [5] appeared in which they determined all spectral sets on \([1, 2)\). However, their proof does not enable one to construct orthonormal systems corresponding to the given spectrum.


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Fix a system of nonintersecting intervals $E_k \subseteq [0, 1]$ such that $|E_k| = b_k$, and set

$$
\varphi_n(t, a, b) = \varphi_n^*(t) = \begin{cases} 
0, & \text{if } t \in E_k, \; 1 \leq k < n, \\
\alpha^2 b_n^{-1}, & \text{if } t \in E_n, \\
-R_n^{-1} a_k, & \text{if } t \in E_k, \; k > n,
\end{cases}
$$

where $R_n = \sum_{k=n+1} ^\infty a_k^2 b_k$ (see [2]), and

\begin{align*}
R_{-1}^1 \alpha a_n & \text{ if } t \in E_k, \; t \in k \in n, \\
0 & \text{ if } t \in E_k, \; k > n.
\end{align*}

We note (see [2]) that

$$
q_n \in L^2 \text{ for } x \in P', \; q_n \in L^2 \text{ for } x \in P' \text{ (} L^2 = M), \; q_n^* \in M.
$$

Let $x_i(k)$ (i = 1, 2, . . . ) be the functions of the Haar system constructed on the $E_k$ and extended by zero to [0, 1]. Set

\begin{align*}
\psi_i(t, a, b) &= \begin{cases} 
\varphi_n, & \text{if } t = \frac{n^2 - n + 2}{2}, \; n = 1, 2, \ldots, \\
\chi^{(n-1)}_{a_k}, & \text{if } t = \frac{n^2 - n + 2}{2} + i, \; 1 \leq i \leq n-1,
\end{cases} \\
\chi^{(n)}_{a_k} & \text{if } t = \frac{n^2 - n + 2}{2} + i, \; 1 \leq i \leq n-1,
\end{align*}

\begin{align*}
\varphi_0(t, a, b) &= -R_n^1 a_k, & \text{if } t \in E_k \\
\{R_0 = \sum_{k=1} ^\infty a_k^2 b_k\}, \\
\chi^{(n)}_{a_k} & \text{if } t = \frac{n^2 - n + 2}{2} + i, \; 1 \leq i \leq n-1,
\end{align*}

Note that $\varphi_0$ also holds for $\varphi_0^*$.

When a function $f$ assumes a constant value on $E_k$, we will denote this value by $f(E_k)$.

**Lemma 1.** The system $\{\varphi_i\}_{i=0} ^\infty$ is orthogonal on [0, 1] and closed in $L^2[0, 1]$ for $\alpha \notin P'$.

**Proof.** Note that

$$
\int _0 ^1 \varphi_n(t) \chi^{(n)}_{a_k} (t) dt = \int _E \varphi_n(t) \chi^{(n)}_{a_k} (t) dt = 0.
$$

Moreover,

$$
\int _0 ^1 \chi^{(n)}_{a_k} (t) \chi^{(n)}_{a_k} (t) dt = \int _E \chi^{(n)}_{a_k} (t) \chi^{(n)}_{a_k} (t) dt = 0
$$

if $(k, l) \neq (k_1, l_1)$. Therefore, in order to prove orthogonality it suffices to verify that $(\varphi_m, \varphi_n) = 0$ for $m = n, m = 0, 1, \ldots, n = 0, 1, \ldots$. Since for $m < n$

$$
\varphi_m(t) \varphi_n(t) = \begin{cases} 
0, & \text{if } t \in E_k, \; 1 \leq k < n, \\
R_n^2 b_n^2, & \text{if } t \in E_n, \\
-R_n^1 a_k, & \text{if } t \in E_k, \; k > n,
\end{cases}
$$