If we replace \( L_0(\ln^2 n) \) on the right by \( L_0(n) \), the inequality (39) is strengthened. If we then replace \( C_4(v) \) on the right by the larger quantity \( C_5(v) \), we can arrange for this strengthened inequality to hold for all \( n \geq n(v) \):

\[
R_n[f;[a,b]] \leq C_5(v) \frac{1}{n} L_0(n) \inf_{(b-a)\kappa_n \leq \lambda < b-a} \left\{ \omega(\lambda, f) + M(f) \frac{1}{n} + M(f) \frac{1}{n} \ln \frac{b-a}{\lambda} \right\}.
\]

(40)

Let us simplify the right-hand side in this inequality. If \( \lambda \geq (b-a)e^{-1} \), then

\[
M(f) \frac{1}{n} \leq M(f) \leq \omega((b-a), f) \leq \omega(\varepsilon \lambda, f) \leq 3\omega(\lambda, f).
\]

If \( \lambda < (b-a)e^{-1} \), then

\[
M(f) n^{-1} \leq M(f) n^{-1} \ln \frac{b-a}{\lambda}.
\]

We thus find from (40) the inequality (38) with the constant \( 4C_5(v) \). Theorem 3 is proved.

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LITERATURE CITED


A LOWER BOUND FOR THE ERROR OF A FORMULA FOR APPROXIMATE SUMMATION IN THE CLASS \( E_{S,p}(C) \)

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The error of a formula for approximate summation in the class \( E_{S,p}(C) \) over an arbitrary mesh containing \( p \) base-points is shown to be not less than \( C_1 \ln^p p \). This estimate has the same order as the error of the optimal parallelepiped mesh in this class.

One possible approach in problems of approximate integration is as follows: The integral is first evaluated by means of a quadrature formula, utilizing a reasonably large number of base-points and thus involving a small error; then the sum obtained is evaluated approximately with respect to a smaller number of base-points. The problem of approximate integration is thus reduced to a problem of approximate summation.

An approximate summation formula for functions of the class \( E_{S,p}(C) \) was obtained by Korobov in [1] with the aid of optimal coefficients. The formula has an error \( \mathcal{O}(p^{-1}\ln^p p) \) and cannot be improved on parallelepiped meshes. Here we shall show that the error of the approximate summation formula with \( p \) base-points over an arbitrary mesh, in the class

Es,p(C), is not less than \(C_1p^{-1}\ln^8 p\); hence, it follows that parallelepiped meshes are order-wise optimal in the class \(E_{s,p}(C)\).

Our estimate is obtained by means of the so-called linear-algebraic method. The method was originally used by the present author for obtaining a similar result in the class \(E^p_s\) [2].

We shall first prove a lemma, which is of independent interest.

**Lemma.** Let \(L_k\) be a \(k\)-dimensional subspace in \(n\)-dimensional space \(R^n\). There exist \(\vec{e} = (e_1, \ldots, e_n) \in L_k\) and \(j_0\) such that

\[|c_j| \geq \frac{k}{n}, \quad |c_j| \leq \frac{n-k}{n}, \quad j \neq j_0.\]

**Proof.** Let \(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\) be a basis in \(L_k\). Further, let \(A = \{a_{ij}\}\) be an \(n \times n\) matrix such that its first \(k\) rows are the same as \(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\); i.e., the elements \(a_{ij}, 1 \leq i \leq k, 1 \leq j \leq n,\) are given, while all the remaining elements are equal to \(+1\) or \(-1\), the signs being chosen in such a way that the absolute value of \(\det A\) is maximized. It can easily be seen that the following equation holds for this matrix, for all \(i \geq k + 1\)

\[\det A = \sum_{j=1}^{n} |A_{ij}|,\]

where \(A_{ij}\) is the cofactor of the element \(a_{ij}\).

Let \(\vec{a}_i, 1 \leq i \leq n,\) be the rows of the matrix. Consider in \(R_n\) the \(n\) equations

\[\sum_{i=1}^{n} a_{ij} \vec{a}_i = \vec{e}_j, \quad j = 1, 2, \ldots, n, \quad \vec{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0).\]

Then, \(a_{ij} = A_{ij}/\det A;\) we denote

\[\vec{b}_j = \sum_{i=k+1}^{n} a_{ij} \vec{a}_i, \quad \vec{b}_j = (b_{j1}, b_{j2}, \ldots, b_{jn}),\]

\[\|\vec{b}_j\| = \max_i |b_{ji}|.\]

We have

\[\sum_{j=1}^{n} \|\vec{b}_j\| \leq \sum_{i=k+1}^{n} |a_{ij}| = \sum_{i=k+1}^{n} \|\vec{a}_i\| = \frac{1}{\det A} \sum_{i=k+1}^{n} \sum_{j=1}^{n} |A_{ij}|.\]

Recalling (1), we obtain \(\sum_{j=1}^{n} \|\vec{b}_j\| \leq n - k.\) Hence, a number \(j_0\) exists such that \(\|\vec{b}_{j_0}\| \leq (n-k)/n.\)

Hence, the vector \(\vec{c}_{j_0} = \vec{e}_{j_0} - \vec{b}_{j_0} \in L_k\) and moreover,

\[|\vec{c}_{j_0}| \geq |c_{j_0}| \geq 1 - |\vec{b}_{j_0}| \geq k/n,\]

\[|c_{j_0}| \leq \frac{n-k}{n}, \quad i \neq j_0,\]

where \(\vec{c}_i = (c_{i1}, c_{i2}, \ldots, c_{in}),\) QED.

Before we state our theorem, let us recall the definition of the class \(E_{s,p}(C)\). We shall say that the function \(f(x_1, \ldots, x_s)\), defined at points \((z_1/p, \ldots, z_s/p), z_i = 0, 1, \ldots, p - 1; i = 1, 2, \ldots, s,\) belongs to the class \(E_{s,p}(C)\) if, at each of these points, we have the equation

\[f(z_1/p, \ldots, z_s/p) = \sum_{m_1, \ldots, m_s = -p}^{p} c_p (m_1, \ldots, m_s) e^{2\pi i (m_1z_1 + \ldots + m_sz_s)/p},\]

\[p_1 = [(p - 1)/2], \quad p_2 = [p/2],\]

where, for the finite Fourier coefficients \(c_p (m_1, \ldots, m_s)\) of the function (we shall henceforth omit the word "finite" and the index \(p\)), we have the estimate

\[|c_p (m_1, \ldots, m_s)| \leq (m_1, \ldots, m_s)^{-1},\]

where \(\bar{m} = \max(1, |m|).\)

**Theorem.** No matter what the base-points \((x_1^{(i)}, \ldots, x_s^{(i)}), i = 1, 2, \ldots, p\) with \(p \geq 2,\) of the approximate summation formula, a function \(f(x_1, \ldots, x_s) \in E_{s,p}(C)\) exists, such that it vanishes at these base-points, and such that