A NONLINEAR DIFFERENTIAL EVASION GAME

N. Satimov

The evasion problem for nonlinear differential games with a target set which is a linear subspace is considered. A sufficient condition for the possibility of avoidance of contact with all points not belonging to the target set is obtained. Examples are considered. This article borders on the investigations of L. S. Pontryagin, E. F. Mishchenko, and B. N. Pshenichnyi.

1. Let the movement of a point \( z \) of the \( n \)-dimensional Euclidean vector space \( \mathbb{R}^n \) be described by the equation

\[
\dot{z} = F(z, u, v),
\]

where \( u \) and \( v \) are control parameters, \( u \) being the parameter of pursuit and \( v \) being the parameter of evasion, \( u \in P, \ v \in Q, \ P \) and \( Q \) are nonempty compact subsets of the \( p \)-dimensional and the \( q \)-dimensional Euclidean spaces \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively; \( F(z, u, v) \) is defined on the set \( \mathbb{R}^n \times P \times Q \) and is a function continuous with respect to \( z, u, \) and \( v \) on this set. Further, the target set \( M \) is selected in \( \mathbb{R}^n \). The differential game is defined by these data.

We will say that the avoidance of contact of \( z_0 \in \mathbb{R}^n \setminus M \) with the set \( M \) is possible if for an arbitrary measurable variation \( u(t) \) of the parameter \( u \in P \) there exists a measurable variation \( v(t) \) of the parameter \( v \in Q \), such that every solution of the equation

\[
\dot{z} = F(z, u(t), v(t)), \quad z(0) = z_0,
\]

is defined for all \( t > 0 \) and none of them belongs to the set \( M \) for any value of the time \( t \) such that \( 0 < t < \infty \). Moreover, for finding the value \( v(t) \) of the parameter \( v \) at every moment of time \( t > 0 \) we are permitted to use only the values \( z(t) \) and \( u(t) \) of the vector of the phase variable \( z \) and the parameter \( u \) at the same moment of time \( t \).

If we can avoid the contact of an arbitrary point \( z_0 \in \mathbb{R}^n \setminus M \), with the set \( M \), then we will say that evasion is possible in the game (1).

The evasion problem for linear, quasilinear, and nonlinear games has been studied in [1-9].

In the sequel positive numbers which depend only on the game (1) and are independent of both the initial point \( z_0 \) and the controls \( u(t) \) and \( v(t) \) will be called constants.

2. In this section we will formulate conditions to be imposed on the game (1) and the basic results of the present article.

**Condition 1.** The function \( F(x, u, v) \) satisfies the Lipschitz condition

\[
| F(z_1, u, v) - F(z_2, u, v) | \leq c_1 | z_1 - z_2 | \tag{3}
\]

for arbitrary \( z_1, z_2 \in \mathbb{R}^n \), \( u \in P \), and \( v \in Q \). In inequality (3) the number \( c_1 \) is a constant.

**Condition 2.** The target set \( M \) is a linear subspace \( \mathbb{R}^n \).

We will denote the orthogonal complement of \( M \) in \( \mathbb{R}^n \) by \( L \), an arbitrary subspace of the space \( L \) by \( W \), the operator of the orthogonal projection from \( \mathbb{R}^n \) onto \( W \) by \( \pi \), the unit sphere in \( L \) with the center at the point \( 0 \in \mathbb{R}^n \) by \( S \), and the scalar product of vectors \( z_1, z_2 \in \mathbb{R}^n \) by \( (z_1, z_2) \).

**Condition 3.** There exist a natural number \( k \) and a subspace \( W \subset L \) such that (see [9]).

a) the function \( \psi = \pi F(z, u, v) \) is independent of the variables \( u \) and \( v \) and is continuous and continuously differentiable with respect to \( z \) for all \( z \in \mathbb{R}^n \). Analogously, the functions \( \varphi_i = \pi \frac{\partial F}{\partial z} F(z, u, v) \), \( i = 1, 2, \ldots, k \), \( \varphi_k = \varphi \), are independent of the variables \( u \) and \( v \) and are continuous and continuously differentiable with respect to \( z \) for all \( z \in \mathbb{R}^n \);

b) the function \( \varphi_{k-1} = \pi \frac{\partial F}{\partial z} F(z, u, v) \) can be represented in the form

\[
\varphi_{k-1} = \psi_1(z) + \psi_2(u, v), \tag{4}
\]

where the functions \( \psi_1(z) \) and \( \psi_2(u, v) \) are continuous with respect to \( z \) and \( (u, v) \), respectively, and the function \( \psi_1(z) \) satisfies the Hölder condition

\[
| \psi_1(z_1) - \psi_1(z_2) | \leq c_2 | z_1 - z_2 |^{m_1}, \tag{5}
\]

where \( c_2 \) and \( m_1 \) are constants.

Theorems 1 and 2 formulated below are the basic results of this article.

**THEOREM 1.** Let the conditions 1-3 be satisfied. If, moreover, \( \dim W = 2 \) and the set \( R = \bigcap_{u \in P} \psi_3(u, Q) \) contains an interior (with respect to \( W \)) point, then evasion is possible in the game (1). Moreover, the process of avoidance of contact of a point \( z_0 \in \mathbb{R}^n \setminus M \) with the set \( M \) can be controlled such that the estimate

\[
\xi(t) \geq \begin{cases} \frac{c_{\xi}^2}{\left(1 + \eta(t)\right)^m} \varepsilon, & \varepsilon_0 < \varepsilon, \\ \frac{c_{\xi}^2}{\left(1 + \eta(t)\right)^m} \varepsilon, & \varepsilon_0 \geq \varepsilon, \end{cases}
\]

where \( c, \varepsilon, \) and \( m \) are constants, holds for the distances \( \xi(t) \) and \( \eta(t) \) of the point \( z(t) \) from the subspaces \( M \) and \( L \) respectively for all \( t > 0 \).

**THEOREM 2.** Let the conditions 1-3 be fulfilled. If, moreover, \( \dim W > k + 1 \) and there exists a vector \( l \in W \) such that one of the numbers \( N_1 \) and \( N_2 \), where

\[
N_1 = \min \max \min_{\psi \in S \psi \in W} (\psi, \psi_3(u, v) - l), \quad N_2 = \min \max \min_{\psi \in S \psi \in W} (\psi, \psi_3(u, v) - l), \tag{7}
\]

...