MOTION OF A CIRCULAR CYLINDER NEAR A VERTICAL WALL

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We consider the arbitrary motion of a circular cylinder in an ideal fluid near a vertical wall. This problem is usually solved in the approximate formulation with a degree of error which is difficult to assess, increasing with approach of the cylinder to the wall [1, 2]. The exact solution has previously been carried out only for the case of purely circulatory flow about the cylinder [3].

§1. Let us consider a cylinder of radius $a$, moving in an ideal fluid which is at rest at infinity. Let the motion of the cylinder be characterized by the velocity components $U$ and $V$. We take the coordinate origin at the center of the cylinder (Fig. 1), then the complex potential of the absolute flow is represented by the expression

$$W = -\frac{a^2}{z} (U + iV) + F(z). \quad (1.1)$$

Here $F(z) = Uf_1(z) + Vf_2(z)$ is a function which is analytic outside the circle in the semiplane bounded by the wall. The function $F$ satisfies the conditions:

1. on the circumference of the circle

$$\text{Im} \ F = 0; \quad (1.2)$$

2. on the wall

$$\text{Im} \ f_1(z) = \text{Im} \frac{a^2}{z}, \quad \text{Im} \ f_2(z) = \text{Re} \frac{a^2}{z}; \quad (1.3)$$

3. at infinity

$$\frac{dF}{dz} \bigg|_{z \to \infty} = 0, \quad F \big|_{z \to \infty} = 0. \quad (1.4)$$

Here $b$ is the distance from the center of the circle to the wall, varying in time (Fig. 1).

Let us map the considered region onto an annulus (Fig. 2). The function which realizes this mapping has the form

$$z_1 = \frac{s - z \xi}{z - n\xi}, \quad n = \frac{b + \sqrt{b^2 - a^2}}{a}. \quad (1.5)$$

In this case the circle of radius $a$ transforms to a unit circle, and the wall transforms to a circle of radius $n$.

We further use the transformation $\xi = -i \ln z_1$ to develop the annulus into a strip segment (Fig. 3) of width $h = \ln n$, bounded by the straight lines $\xi = 0$ and $\xi = 2\pi$. In the $\xi$ plane condition (1.2) takes the form

$$\text{Im} \ F(\xi) = 0, \quad (1.5)$$

and conditions (1.3) are written as

$$\text{Im} f_1(\xi) \big|_{\xi = -h} = -a \sin \frac{\xi}{b} \frac{\sin \pi \xi}{\cosh \pi \xi}, \quad (1.6)$$

$$\text{Im} f_2(\xi) \big|_{\xi = -h} = a \frac{\sin \frac{\pi \xi}{b}}{\cosh \frac{\pi \xi}{b} \cos \frac{\pi \xi}{b}}. \quad (1.7)$$

From (1.6) and (1.7) it follows that $F(\xi)$ has period $2\pi$, which permits its analytic continuation onto an infinite strip. Moreover, condition (1.5) additionally permits the function $F$ to be continued also in the portion $\text{Im} \ \xi = h$ of the upper semiplane.

The determination of the complex potential $W$ reduces to finding the function $F$ which is holomorphic over the entire strip and $\text{Im} \ \xi \leq h$. From the analytic continuation conditions we have

$$v_1(-) = -v_1(+), \quad u_1(-) = u_1(+). \quad (1.8)$$

Here $v_1(-)$, $v_1(+)$ and $u_1(-)$, $u_1(+)$ are, respectively, the values of the imaginary and real parts of $f_1(\xi)$ at the lower and upper boundaries of the region. Then the Schwartz integral for the strip takes the form

$$f_1(\xi) = \frac{1}{2k} \int_{-\infty}^{\infty} u \text{sch} \frac{\pi (\xi - \xi)}{2h} \ d\xi, \quad (2.1)$$

or

$$f_1(\xi) = \frac{1}{2k} \text{sh} \frac{\pi \xi}{2h} \int_{-\infty}^{\infty} \text{sch} \frac{\pi \xi}{2h} \text{sch} \frac{\pi (\xi - \xi)}{2h} \ d\xi \quad (2.2)$$

Here $v(-)$ is defined by (1.8). The integral in (2.2) cannot be expressed in terms of elementary functions, in view of which we express the function $f_1(\xi)$ in the form of a series.
We see from (1.6) that \( v(-) \) is an odd trigonometric function relative to \( \xi \). Then \( u \) is an even function which may be represented by a Fourier series of the form

\[
u = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\xi.
\]

Thus

\[
v(-) = \text{Im} f_1(\xi)|_{\xi=-h} = \sum_{k=1}^{\infty} A_k \text{th} kh \sin k\xi.
\]

On the other hand, expanding (1.6) in a Fourier series, we have

\[
v(-) = 2a \sin h \sum_{k=1}^{\infty} \frac{(-1)^k}{\sin k\xi} e^{2kh} \sin k\xi.
\]

Comparing (2.4) and (2.5), we obtain

\[
A_k = 2a \sin h \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \right).
\]

Consequently,

\[
f_1(\xi) = \frac{A_0}{2} + 2a \sin h \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \right) \cos k\xi.
\]

Similarly, we have

\[
f_2(\xi) = \frac{A_0}{2} + 2a \sin h \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \right) \sin k\xi.
\]

Then finally the complex potential \( W \) exact to within a constant is represented as

\[
W_1 = -\frac{a^2}{2} U + 2aU \sin h \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \right) \cos k\xi,
\]

\[
W = W_1 + W_2,
\]

\[
W_2 = -\frac{a^2}{2} iV + 2aV \sin h \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \right) \sin k\xi.
\]

Here and in what follows the subscript \( l \) corresponds to motion toward the wall, \( 2 \) corresponds to motion along the wall

\[
\zeta = -i \ln \frac{a - nz}{z - na}.
\]

The series in (2.8) and (2.9) converge for \( h > 0 \), i.e., for any \( b > a \). The potential of purely circulatory flow about the cylinder is represented in the form

\[
W_2 = \Gamma \frac{\text{ln} \left( \frac{a}{a - z} \right)}{2\pi z},
\]

For sufficient distance of the cylinder from wall \( (b = \infty) \) the complex potential \( W = W_1 + W_2 + W_3 \) takes the form

\[
W|_{b=\infty} = \frac{1}{2\pi} \ln \frac{\Gamma}{\pi} - \frac{a}{z} (U + iV),
\]

i.e., it is the potential of motion of the cylinder in an unbounded fluid.

Using Eqs. (2.8) and (2.9), we determine the velocities:

1. At the wall \( v_{1x} = v_{2x} = 0 \)

\[
\frac{d\zeta}{dz} = -\frac{2a \sin h}{\xi - a^2 + y^2} \xi;
\]

\[
v_{1y} = 2a \sin h \left( \frac{\xi - a^2 + y^2}{\xi + a^2 - y^2} \right) U +
\]

\[
-\frac{4a^2 \sin h \xi}{b^2 - a^2 + y^2} \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \text{th} kh \sin k\xi;
\]

\[
v_{2y} = -\frac{a \sin \theta}{b^2 - a^2 - y^2} V +
\]

\[
+ \frac{4a^2 \sin h \xi}{b^2 - a^2 - y^2} \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \text{th} kh \cos k\xi;
\]

\[
tg \xi = \frac{2y \sin h}{b^2 - a^2 - y^2}.
\]

2. On the circumference of a circle of radius \( a \) in polar coordinates

\[
\frac{d\zeta}{dz} = -\frac{\sin \theta}{a} \cos \theta,
\]

\[
v_{1x} = U \cos \theta,
\]

\[
v_{1y} = U \sin \theta - \frac{2U \sin h \xi}{b^2 - a^2 - y^2} \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \sin k\xi;
\]

\[
v_{2x} = V \sin \theta,
\]

\[
v_{2y} = -V \cos \theta + \frac{2V \sin h \xi}{b^2 - a^2 - y^2} \sum_{k=1}^{\infty} \left( \frac{-1)^k}{\sin k\xi} e^{2kh} \cos k\xi;
\]

\[
tg \xi = \frac{\sin \theta \sin h}{a \cos \theta \sin h - 1}.
\]