A CHARACTERIZATION OF A FINITE SIMPLE HALL–JANKO GROUP

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A finite simple Hall–Janko group in the class of simple groups is characterized by the structure of the centralizer of a central involution.

In a series of papers by Janko, Thompson, and Ward [1–3], results were obtained which can be summarized as follows.

Let $G$ be a finite simple group which satisfies the following conditions:

a) the Sylow 2-subgroups of $G$ are Abelian;

b) there exists an involution $r$ in $G$ such that $C_G(r) = \langle r \rangle \times F$, $F \cong PSL(2, q)$, $q \geq 3$.

Then $G$ is a group of Ree type or $G$ is isomorphic to the Janko group of order 175,560. As reported by V. D. Mazurov, condition a) follows from b). Thus the finite simple groups are determined in which the centralizer of some involution is the direct product of a group of order 2 and $PSL(2, q)$.

On the other hand, in the new simple Hall–Janko group of order 604,800 [4], there is an involution $r$ such that $C_G(r) = V \times F$, $V$ is an elementary Abelian group of order 4, $F \cong PSL(2, 5)$ (where $C_G(r) = C_G(v)$ for all $v \in V^\#$). This makes it interesting to study finite groups in which the centralizer of some involution is the direct product of the four group and $PSL(2, q)$.

In the present paper the following theorem is proved:

**THEOREM.** Let $G$ be a finite simple group which contains a subgroup $H$ such that $H = V \times F$, where $V$ is an elementary Abelian group of order 4, $F$ is isomorphic to $PSL(2, q)$, $q$ is odd, and $H = C_G(v)$ for all $v \in V^\#$. Then $G$ is isomorphic to the Hall–Janko group.

In the proof of the theorem, we shall show that a Sylow 2-subgroup of $G$ is isomorphic to a Sylow 2-subgroup of the Hall–Janko group. Then by a result of Gorenstein and Harada [5], $G$ is isomorphic to the Hall–Janko group or the third Janko group [4]. But in the latter case the conditions of the theorem do not hold.

Throughout this paper $T$ denotes a Sylow 2-subgroup of $F$ and $A = VT$.

$S$ is a Sylow 2-subgroup of $G$ containing $A$. $C^\#$ denotes the set of nonidentity elements of $C$.

The normalizer and centralizer of $L$ in the whole group $G$ will be denoted by $N(L)$ and $C(L)$, respectively.

The rest of the notation is standard.

**LEMMA 1.** For any $g \in S \setminus A$ we have $V \cap Vg = 1$.

**Proof.** Let us first show that $A$ is a Sylow 2-subgroup of $N(F)$. Indeed, otherwise there exists a subgroup $R \subseteq N(F)$, such that $[R : A] = 2$. Since $C(F) \subseteq N(F)$ then $R \subseteq N(C(F) \cap A)$. Taking into account $C(F) \cap A = V$, we obtain $R \subseteq N(V)$. Then $R$ centralizes some element of $V^\#$. Contradiction.
Now assume that \( v \in V \cap V^\# \), \( v \neq 1 \), \( g \in S \). Then \( Fg = C(v) = H \). From the structure of \( H \) it follows that \( Fg = F \). Hence \( g \in N_G(F) = A \). This proves the lemma.

**Lemma 2.** Let \( g \in N_S(A) \setminus A \), \( g^2 \in A \). Then \( \langle A, g \rangle = V \cup \langle g_1 \rangle \), where \( |g_1| = 2 \).

**Proof.** As a preliminary, let us show that \( |T| = 4 \). By virtue of the main result of [6], we can assume that \( A \) is not a Sylow 2-subgroup of \( G \). Let \( g \in N_S(A) \setminus A \). Then \( g \in N(VZ(T)) \). If \( |T| > 4 \), then \( |Z(T)| = 2 \). Since \( V, V^\# \subset VZ(T), V \cap V^\# \neq 1 \), which contradicts Lemma 1. Thus \( |T| = 4 \).

Since \( V \cap V^\# = 1 \) for all \( g \in S \setminus A \), for the proof it suffices to find an involution \( g_1 \) in \( \langle A, g \rangle \setminus A \).

Since \( A = V \times V \), then \( g^2 = \frac{v_1}{v_2} v_2 \), \( v_2 \in V \). Then \( g^2 = g_1^2 \), i.e., \( g_1 = \frac{v_1}{v_2} \). Hence \( v_1 = v_2 \), \( g^2 = v_1^2 \). Then \( v_1 g = g_1 v_1 \), i.e., \( g_1 = v_1^{-1} g_1 \) is an involution in \( \langle A, g \rangle \setminus A \). This proves the lemma.

Lemma 2 states that all subgroups of \( S \) which contain \( A \) as a subgroup of index 2 are wreath products of the four group \( V \) and a group of second order. This property basically determines the structure of \( S \).

Since by [7] \( A \) is not a Sylow 2-subgroup of \( G \), \( |S| \leq 2^6 \). If \( |S| = 2^5 \), then by Lemma 2, \( S \) is the wreath product of the four group and a group of order 2. As shown by Harada ([8], Lemma 18), \( G \) then possesses a subgroup of index 2, which contradicts the simplicity of the group. Thus \( |S| \neq 2^6 \).

**Lemma 3.** The following statements hold:

a) \( S \) has a unique invariant four subgroup \( T \);

b) the elements of \( A \setminus T \) are divided into three classes of conjugate elements in \( S \) with representatives \( v_1, v_2, v_3 \), where \( \{v_1, v_2, v_3\} = V^\# \);

c) \( N(A) \subseteq N(T) \).

**Proof.** It is not difficult to see that there exists an invariant elementary Abelian subgroup \( U \) of order 4 in \( S \). Then \( |S : C_S(U)| \leq 2 \) and consequently \( C_S(U) \cap V \neq 1 \). Hence \( U \subseteq C_S(v) = A \) for some \( v \in V^\# \). Since the order of a Sylow 2-subgroup of \( C(v) \) for \( v \in V^\# \) is equal to \( 2^3 \) and the order of a Sylow 2-subgroup of \( C(u) \) for \( u \in U^\# \) is greater than \( 2^4 \), no element of \( U^\# \) is conjugate in \( G \) to an element of \( U^\# \). In addition, taking subgroups \( S_1 \) and \( S_2 \) such that \( A \subseteq S_1 \subseteq S_2 \), and \( |S_1 : A| = |S_2 : S_1| = 2 \), we obtain that \( A \) is a characteristic subgroup of \( S_1 \) and \( A \subseteq S_2 \). Therefore by virtue of Lemma 1, \( U \) is equal to the difference of \( A \) and the set of elements of \( A \) which are conjugate in \( S_2 \) to elements of \( V^\# \). \( U \) is invariant in \( N(A) \).

Let \( x \) be a 3-element of \( N_P(T) \setminus C_P(T) \). Then \( [A, x] = T \). On the other hand, \( [A, x] = [VU, x] = [U, x] \subseteq U \). Hence \( U = T \). This proves the lemma.

In Lemmas 4 and 5, \( |S| = 2^6 \).

**Lemma 4.** \( S/A \) is noncyclic.

**Proof.** Assume the contrary. Fong (see [8]) showed that if \( G \) is a simple group and the order of a Sylow 2-subgroup \( S \) of \( G \) is equal to \( 2^s \), then \( S \) is isomorphic to a dihedral group, a semidihedral group, a Sylow 2-subgroup of the Mathieu group \( M_{12} \), the direct product of the four group and a semidihedral group, or \( S \) is of exponent 4. By virtue of Lemma 2, all cases except the last are impossible.

Thus the exponent of \( S \) is equal to 4. Since \( S/A \) is cyclic, \( S \) is a splitting extension of \( A \), \( S = A \rtimes \langle t \rangle \).

By a theorem of Gaschütz (see [9], Theorem 15.8.6), \( N(A) \) is a splitting extension of \( A \), \( N(A) = A \rtimes \langle K \rtimes \langle t \rangle \rangle \).

Since \( C(A) = A \), \( K \subseteq N(A) \subseteq N(T) \) and \( N(A)/A \) is a subgroup of \( GL(4, 2) \), either \( |K| = 3 \) or \( |K| = 9 \), and \( K \) is an elementary Abelian group. We can assume that \( x \in K \). Then \( K = C_K(T) \times \langle x \rangle \).

Since \( C_K(T) \) is \( t \)-admissible, applying Maschke's theorem, we obtain \( [K, t^2] = 1 \), in particular, \( [x, t^2] = 1 \). From \( x \in C(V) \) for \( v_1 \), \( v_2 \in V^\# \) we obtain \( v_1 t^2 = (v_1 t^2)^x = (v_1 t^2)^x \) and \( v_2 t^2 = (v_2 t^2)^x = (v_2 t^2)^x \). Since \( A = \langle V, V^\# \rangle = \langle v_1, v_2, v_3, v_4 \rangle \), then \( [A, x] = 1 \), a contradiction. This proves the lemma.

**Lemma 5.** \( S \) is isomorphic to a Sylow 2-subgroup of \( PSL(3, 4) \).

**Proof.** A Sylow 2-subgroup \( S \) of \( PSL(3, 4) \) is generated by the involutions \( z_1, z_2, a_1, a_2, b_1, b_2 \), where \( S = \langle z_1 \rangle \times \langle z_2 \rangle \times \langle a_1 \rangle \times \langle a_2 \rangle = \langle b_1 \rangle \times \langle b_2 \rangle \), \( |z_1| = |b_1| = 4 \), \( |a_1| = |a_2| = 2 \), \( |z_2| = |b_2| = 2 \), and \( |z_2| = 2 \times 2 \times 2 \). Since \( S/A \) is noncyclic, there exist two subgroups \( S_1 \) and \( S_2 \) of \( S \) such that \( S_1 : A = S_2 : A = 2 \). By Lemma 2, \( S_1 = V \cup \langle b_1 \rangle \) and \( S_2 = V \cup \langle b_2 \rangle \), where \( b_1 \) and \( b_2 \) are involutions. Since \( Z(S_1) \) and \( Z(S_2) \) are invariant four groups in \( S \), by Lemma 3 (part a) they coincide with \( T \) and \( T \) coincides with the center of \( S \). Setting \( V = \{1, v_1 v_2, v_3\} \), we obtain \( Z(S_1) = \{1, v_1 v_2, v_2 v_3, v_1 v_3\} = Z(S_2) = \{1, v_1 v_2, v_2 v_3, v_1 v_3\} \). In addition, \( v_1 v_2^i = v_2 v_3^i \) (\( i = 1, 2, 3 \)), or else \( v_1 v_2^i = v_1 \), but \( b_1, b_2 \in S \setminus A \), which contradicts Lemma 1.