

CONFIGURATION SPACES OF BRICARD OCTAHEDRA

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UDC 514

We study the space of positions of a deformable octahedron and give a complete topological characterization of the structure of this space.

The existence of deformable octahedra was first discovered by Bricard [1]. He also gave their complete classification, distinguishing three different types by their structure. Since then several more methods of finding deformable octahedra were given with the Bricard classes described from different points of view [2-4]. On the basis of the formulas offered below, the known classes of deformable octahedra could also be distinguished. However, we take up the problem of the structure of the configuration spaces, or the set of positions, occupied by the octahedron in the course of its deformation.

It turns out that the study of the topological and the algebraic-geometrical structures of such varieties is a sufficiently interesting problem in itself, whose solution allows us to characterize or distinguish anew the types of deformable octahedra. In addition, the following should be kept in mind: the configuration space of any deformable polyhedron is an algebraic variety; therefore, the obtained description of the configuration spaces of deformable octahedra gives an idea of the complex topological situations that will be encountered when trying to describe such varieties.

1. Basic Formulas. We consider a polyhedron consisting of the sides (\(M_1M_2M_3\)), (\(M_3M_2M_4\)), (\(M_5M_1M_4\)), (\(M_5M_2M_5\)). We call such a polyhedron a "cap" \(M_1M_2M_3M_5\) with vertex \(M_3\) (Fig. 1). To describe the nontrivial deformations of the "cap," we fix in space a plane \(\Pi\) and on it points \(M_1, M_3\). We will also assume that the point \(M_2\) always remains on the plane \(\Pi\). When the point \(M_2\) moves on the plane, the "cap" bends in a certain way. The set of all possible positions of such a "cap" will be called its configuration space. To investigate the configuration spaces of the "cap," we introduce the following parameters: \(x = |M_1M_2|^2\), \(y_1\) — the square of the distance from point \(M_4\) to the plane \(\Pi\), and \(y_2\) — the square of the distance from point \(M_5\) to the plane \(\Pi\). They are related by the following equations:

\[
y_1 = \frac{-2ab (x - (a^2 + b^2 - 2ab \cos (\alpha_1 + \beta_1)))(x - (a^2 + b^2 - 2ab \cos (\alpha_2 - \beta_2)))}{c_1 (x - (a + b)^2)(x - (a - b)^2)}, \tag{1}
\]

\[
y_2 = \frac{-2ab (x - (a^2 + b^2 - 2ab \cos (\alpha_1 + \beta_1)))(x - (a^2 + b^2 - 2ab \cos (\alpha_2 - \beta_2)))}{c_2 (x - (a + b)^2)(x - (a - b)^2)}, \tag{2}
\]

where \(a = |M_1M_2|; b = |M_1M_3|; c_1 = |M_2M_4|; c_2 = |M_3M_5|; \alpha_1 = M_1M_2M_4; \beta_1 = M_1M_3M_5; \alpha_2 = M_2M_3M_5; \beta_2 = M_1M_3M_5.

Geometrically, the numerators of Eqs. (1) and (2) are the volumes of the tetrahedra \(M_1M_2M_3M_4\) and \(M_1M_2M_3M_5\), respectively, while the denominators are the area of the triangle \(M_1M_2M_3\).

Below we will consider only the nondegenerate deformations, i.e., those for which as parameter \(x\) varies so do parameters \(y_1\) and \(y_2\). In addition, we introduce two more characteristics of the "cap," \(\omega_1\) and \(\omega_2\). The plane \(\Pi\) divides the space into two subspaces \(F_1\) and \(F_2\). We set

\[
\omega_1 = \begin{cases} +1, & \text{if the point } M_4 \in F_1, \\ 0, & \text{if the point } M_4 \in \Pi, \\ -1, & \text{if the point } M_4 \in F_2 \end{cases}, \quad \omega_2 = \begin{cases} +1, & \text{if the point } M_5 \in F_1, \\ 0, & \text{if the point } M_5 \in \Pi, \\ -1, & \text{if the point } M_5 \in F_2 \end{cases}
\]

The triplets of numbers \((x, \omega_1, \omega_2)\) completely define the position of the "cap" in the space. We consider the deformation of the "cap." Admissible values of \(x\) are defined to be those to which a real "cap" corresponds. The following is obvious.

**Lemma.** The set of admissible values coincide with the set \(G = \{x_{11}, x_{12}\} \cap [x_{21}, x_{22}]\), where \(x_{11}, x_{12}\) and \(x_{21}, x_{22}\) are the roots of the numerators of Eqs. (1) and (2).

We have five possible cases (Fig. 2) for arranging segments \([x_{11}, x_{12}]\) and \([x_{21}, x_{22}]\) on a straight line. In the first case the configuration space is either empty or consists of one point, that is, either such a "cap" does not exist at all or it is certainly not deformable.

In the second case the configuration space is homeomorphic to a circle (Fig. 3). Indeed, suppose that for definiteness \(x_{11} < x_{21} < x_{12} < x_{22}\), then the domain of admissible values will be the segment \([x_{21}, x_{12}]\). Let \(x = x_{21}\), then the point \(M_4\) can be placed in two different ways with \(\omega_1 = \pm 1\) and \(\omega = \mp 1\). Obviously, "caps" that are not translatable one into another by motion correspond to the combinations \((x_{21}, \pm 1, 0)\) and \((x_{21}, -1, 0)\). We begin to deform the "cap" by increasing \(x\). Here we have two possibilities for placing the point \(M_4\), one with \(\omega_2 = \pm 1\) and another with \(\omega_2 = -1\).

Thus, for \(x \{(x_{21}, x_{12})\}\), we have four states of the "cap" to which the following characteristics correspond: \((x, \pm 1, \pm 1)\), \((x, \pm 1, -1)\), \((x, -1, \pm 1)\), \((x, -1, -1)\). As the "cap" deforms further, \(x\) will become equal to the root \(x_{12}\) and the point \(M_4\) will lie on the plane, in this case \(\omega_1 = 0\). Thus, the branches \((x, \pm 1, \pm 1)\) and \((x, -1, -1)\), \((x, +1, -1)\) and \((x, -1, -1)\) merge (Fig. 3). We introduce the parameter

\[
\varphi = \begin{cases} 
\frac{\pi}{2} \frac{x - x_{11}}{(x_{12} - x_{11})}, & \text{for } \omega_1 = 1; \omega_2 = 0, l; x \in [x_{11}, x_{12}], \\
+ \frac{\pi}{2} \frac{x - x_{12}}{(x_{12} - x_{11})}, & \text{for } \omega_1 = 0; \omega_2 = 1, l; x \in [x_{12}, x_{13}], \\
\frac{\pi}{2} \frac{x - x_{12}}{(x_{12} - x_{11})}, & \text{for } \omega_1 = -1; \omega_2 = 0, l; x \in [x_{11}, x_{12}], \\
\frac{\pi}{2} \frac{x - x_{11}}{(x_{12} - x_{11})}, & \text{for } \omega_1 = 1; \omega_2 = -1; x \in [x_{21}, x_{12}], \\
\frac{3\pi}{2} \frac{x - x_{12}}{(x_{12} - x_{11})}, & \text{for } \omega_1 = 1, \omega_2 = -1; x \in [x_{21}, x_{12}],
\end{cases}
\]

We have obtained a bijective mapping of the configuration space onto a circle: \((x, \omega_1, \omega_2) \rightarrow (\cos \varphi, \sin \varphi)\).

In the third case the configuration space will be homeomorphic to two circles. Suppose that for definiteness \(x_{21} < x_{11} < x_{12} < x_{22}\), then the domain of admissible values will be the segment \([x_{11}, x_{12}]\). Let \(x = x_{11}\), then the point \(M_4\) will lie on the plane; this corresponds to \(\omega_1 = 0\). The point \(M_6\) here can be placed in the space in two different ways with \(\omega_2 = -1\) and \(\omega_2 = +1\). Thus, we have two possibilities for the initial states of the "caps": \((x_{11}, 0, \pm 1)\) and \((x_{11}, 0, -1)\). As \(x\) increases, point \(M_6\) can be found both in the positive and the negative half-spaces. In this case we will have four possible states of the "cap":