COMPLETENESS AND ORTHOGONALITY OF THE NULL
PLANE OF ONE CLASS OF SOLUTIONS
OF THE RELATIVISTIC WAVE EQUATIONS

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Possible redefinitions of the scalar product are analyzed for relativistic wave fields of
the Klein-Gordon and Dirac types. It is shown that for an entire class of new exact
solutions, for which it was previously not possible to define the usual scalar product
on the $x^0=\text{const}$ plane, it is possible to find a correct scalar product on the null plane
$x^0-x^3=\text{const}$. Orthogonality and completeness relations are proved for this scalar
product. Possible applications of the results are discussed.

Exact solutions of the relativistic wave equations in many new configurations of the external electro-
magnetic field were found in [1-3]. A common feature of all these configurations is that they incorporate
an electric field and a plane wave propagating along this field. Solutions are possible both in the case in
which the electric field is constant and in the case in which it is an arbitrary function of the combination
$x^0-x^3$ (a travelling electric wave). The gauge is chosen such that the potentials of the field of the traveling
wave and the field of the plane wave are functions of $x^0-x^3$ alone.

All the solutions obtained in these fields are such that the corresponding charge densities have non-
integrable singularities at certain points. Because of these singularities, it is not possible to define the
usual scalar product for such solutions in a reasonable manner, so that they cannot be used to calculate
specific effects.

Analysis of the solutions of the classical equations of motion in these fields reveals the physical
sources of these singularities: In these solutions the integrals of motion are such that, as the particles are
accelerated by the electric field to velocities approaching the speed of light, they all tend to lie near a single
plane, which is moving at the speed of light. In quantum mechanics, the result is a divergence of the charge
density on this plane.

Below we show that, despite the singularities of these solutions, they can be normalized in a certain
manner; we prove orthogonality and completeness relations for them, so that these solutions can be used
to calculate specific physical effects.

We will be guided by the following circumstances: We assume that we have found exact solutions of
the relativistic wave equations and that, because of some singularities of the fields and integrals of motion,
the usual scalar product cannot be defined on the $x^0=\text{const}$ plane. Then by using the familiar invariant ex-
pressions for the scalar products on an arbitrary hypersurface [4] we can attempt to find a hypersurface on
which the scalar product of the solutions which have been found would be determined and on which the so-
lutions themselves for this scalar product would form a complete, orthonormal system of functions.

We consider the invariant scalar products of Klein-Gordon and Dirac fields on an arbitrary hyper-
surface $\sigma$:

$$\langle \varphi, \varphi \rangle_{\sigma} = \int d\sigma \varphi^\mu \varphi_\mu,$$

(1)
where \( \gamma_{\mu} = \gamma^0 \left( \frac{i}{\hbar c} \frac{\partial}{\partial x^\mu} - \frac{e}{\hbar c} A_\mu \right) \phi - \gamma \left( \frac{i}{\hbar c} \frac{\partial}{\partial x^0} + \frac{e}{\hbar c} A_0 \right) \psi^* \) for the Klein–Gordon field and \( \chi_{\mu} = \bar{\psi} ω_{\mu} \phi \) for the Dirac field.

We transform to the new curvilinear coordinate
\[
u^\mu = \nu^\mu (x^0, x^1, x^2, x^3), \quad \mu = 0, 1, 2, 3. \tag{2}\]

We assume that \( \nu^\mu (x^0, x^1, x^2, x^3) = \text{const} \) is the desired hypersurface \( \sigma \) on which the scalar product of the fields, (1), is meaningful and is independent of \( \nu^0 \). Then the new scalar product in terms of the variables \( \nu^\mu \) is
\[
\langle \phi, \psi \rangle_{\nu} = \int d\nu \left[ D \left( \begin{array}{c} x^1 \cr x^2 \cr x^3 \cr u^1 u^2 u^3 \end{array} \right) \right] \phi \bar{\psi} + D \left( \begin{array}{c} x^0 \cr x^1 \cr x^2 \cr u^0 u^1 u^2 u^3 \end{array} \right) \psi \bar{\phi} - D \left( \begin{array}{c} x^1 \cr x^2 \cr x^3 \cr u^1 u^2 u^3 \end{array} \right) \phi \bar{\psi} + D \left( \begin{array}{c} x^0 \cr x^1 \cr x^2 \cr u^0 u^1 u^2 u^3 \end{array} \right) \psi \bar{\phi}, \quad d\nu = du^1 du^2 du^3, \tag{3}\]

where \( D \) are the Jacobians of the coordinate transformation. Using these equations, we have succeeded in defining a suitable product (3) for all the solutions in [1–3], and we have proved orthogonality and completeness relations for this product. We turn now to these cases. For each case we write the transformation (2), the potentials for the external electromagnetic field \( A_\mu \) in the new coordinate system, the electric and magnetic fields in a Cartesian coordinate system, the solutions themselves, and the orthogonality and completeness relations.

I. We introduce the new coordinates
\[
u^0 = 2^{-\frac{1}{2}} (x^0 - x^3), \quad \nu^1 = x^1, \quad \nu^2 = x^2, \quad \nu^3 = 2^{-\frac{1}{2}} (x^0 + x^3). \tag{4}\]

In this coordinate system we specify the potentials of the electromagnetic field by \( \tilde{A}_0 = 0, \quad \tilde{A}_j = \tilde{A}_j (\nu^0) \) \((j = 1, 2, 3)\). This choice of potentials corresponds to the following fields in the Cartesian system:
\[
E_x = H_y = 2^{-\frac{1}{2}} \tilde{A}_1, \quad E_y = -H_x = 2^{-\frac{1}{2}} \tilde{A}_3, \quad E_z = \tilde{A}_1, \quad H_z = 0. \tag{5}\]

We see from (5) that \( E_{\mu} H^{\mu} = 0 \), i.e., the fields are orthogonal and represent a combination of the field of a plane wave propagating along the \( x^3 \) axis and a longitudinal electric wave also traveling along \( x^3 \).

Below we denote by \( \tilde{\mathbf{a}}^\mu \) the components of the vectors with respect to the new coordinate system in all cases. In this case the solutions of the Klein–Gordon and Dirac equations are
\[
\tilde{\phi}_B = \frac{2}{(2\pi)^{\frac{3}{2}}} P_{\frac{3}{2}} \exp (-iS), \tag{6}\]
\[
\tilde{\phi}_D = 2^{-\frac{1}{2}} (2\pi)^{-\frac{3}{2}} P_{\frac{3}{2}} \exp (-iS) \frac{(\kappa_0 + \sqrt{2P_3 + \sigma_3 (\sigma_1 P_1 + \sigma_2 P_2)})}{(\kappa_0 - \sqrt{2P_3 + \sigma_3 (\sigma_1 P_1 + \sigma_2 P_2)})} v_c, \tag{7}\]

\[
S = \kappa_1 \nu^1 + \kappa_2 \nu^3 + \kappa_3 \nu^3 + 2^{-1} \int P_{\frac{3}{2}} \left( \kappa_3^2 + P_1^2 + P_2^2 \right) d\nu^0 \quad \quad 
\tilde{P}_j (\nu^0) = \kappa_j - \frac{e}{\hbar c} \tilde{A}_j (\nu^0), \quad \kappa_0 = \frac{mc}{\hbar}, \quad \tilde{\mathbf{a}}^\mu \quad \text{are the eigenvalues of the operators representing the integrals of motion, \( \hat{\kappa}_j = i(\partial / \partial \nu^j) \), \( v_c \) is an arbitrary two-component spinor, and } \kappa = \pm 1 \text{ is the spin quantum number.} \]

The scalar products of solutions (6) and (7) are not defined on the \( x^0 = \text{const} \) plane because of the non-integrable (at certain \( k_j \)) singularity \( P_3^{\frac{3}{2}} \). Let us determine the scalar product on the plane \( x^0 = \text{const} \). In terms of the new variables in (4) it is
\[
\langle \tilde{\phi}, \phi \rangle_{\nu} = 2^{-\frac{1}{2}} \int d\nu \left( \nu_0 + \chi_3 \right). \tag{8}\]

Alternatively, in explicit form, it is
\[
\langle \tilde{\phi}, \phi \rangle_{\nu} = \int d\nu \left[ \frac{e}{\hbar c} \left( \frac{i}{\hbar c} \frac{\partial}{\partial \nu^3} + \frac{e}{\hbar c} \tilde{A}_3 \right) \phi - \phi \left( \frac{i}{\hbar c} \frac{\partial}{\partial \nu^3} + \frac{e}{\hbar c} \tilde{A}_3 \right) \phi^* \right] \tag{9}\]
for the Klein–Gordon equation and
\[
\langle \tilde{\phi}, \phi \rangle_{\nu} = 2^{\frac{3}{2}} \int d\nu \phi^* P_{(-)} \phi; \quad P_{(-)} = \frac{1 - \gamma_0 \gamma_3}{2} \quad \tag{10}\]
for the Dirac equation. A direct calculation shows that such a scalar product is defined for solutions (6) and (7). Furthermore, these solutions satisfy orthogonality and completeness relations for (9) and (10):