DERIVATIVES OF THE RADICALS OF MODULI

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The radicals of moduli are investigated as functors. Particular attention is given over to
the derivatives of the functor of a radical. The function $R^0 r$ turns out to be a hereditary
radical containing the given one. Several relations between the $R^1 r$ are proved. The arti-
cle is concluded with a description of the hereditary radicals of a Prüfer ring. All the
radicals of a Prüfer ring have the form of torsions in an integral domain.

1. Definitions

Throughout the article $A$ denotes an associative ring with identity element.

In every case the module is understood to mean the unitary left $A$-module.

**Definition 1.1.** A class $\%$ of $A$-modules is called radical if $\%$ satisfies the following conditions:

1.1.1. If $A \subseteq \%$, then the homomorphic image of $A$ is a member of $\%$.

1.1.2. Every module $A$ contains a maximal submodule $r(A)$ belonging to $\%$.

1.1.3. $A/r(A)$ is semisimple, i.e., it does not contain nontrivial submodules belonging to $\%$.

If $r(\%)$ is treated as a functor defined on the category of $A$-modules, we obtain the following equivalent
of Definition 1.

**Definition 1.2.** The functor $r: \% \rightarrow \%$, where $\%$ is the category of $A$-modules, is a radical in $\%$ if
it satisfies the following conditions:

1.2.1. For any module $A$ there exists a monomorphism $r(A) \rightarrow A$ which is a proper map of functors
($r$ is a subfunctor of the unit functor).

1.2.2. $rr=r$.

1.2.3. $r(A/r(A))=0$.

**Definition 1.3.** A class $\%$ of $A$-modules is called radical if it satisfies the following conditions:

1.3.1. $0 \in \%$.

1.3.2. If $A \in \%$, and $\subseteq \%$ in the exact sequence $A \rightarrow B \rightarrow C \rightarrow 0$, then $\subseteq \%$.

1.3.3. If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq \cdots$ and $\alpha, A_\alpha \subseteq \%$ for all $\alpha$, then $\cup A_\alpha \subseteq \%$.

**Proposition 1.1.** Definitions 1.1 and 1.3 are equivalent.

**Proof.** Let us suppose that the conditions of Definition 1.3 are fulfilled. We consider the exact se-
quence

$$A \rightarrow B \rightarrow 0 \rightarrow 0,$$

and let $A \in \%$; then 1.1.1 follows from 1.3.2.

Let $U \subseteq A$, $V \subseteq A$ and $U \subseteq \%, V \subseteq \%$; then from the exact sequence $0 \rightarrow U \cap V \rightarrow U \cap V \rightarrow 0$ and the now-verified condition 1.1.1 we get $U+V/V \subseteq \%$, and from the exact sequence $0 \rightarrow V \rightarrow U+V \rightarrow U+U/V \rightarrow 0$ and condition 1.3.2 we get $U+V \subseteq \%$.

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We have thus proved that if submodulus \( U \) and \( V \) of the modulo \( A \) belong to class \( \mathfrak{F} \), their union in \( A \) belongs to class \( \mathfrak{F} \), hence condition 1.1.2 follows from 1.3.3.

Next we let \( r(A) \) denote the maximal submodulo of \( A \) contained in \( \mathfrak{F} \). We investigate the exact sequence

\[
0 \to r(A) \to A \to A/r(A) \to 0
\]

and the sequence induced by it

\[
0 \to r(A) \to B \to r(A/r(A)) \to 0,
\]

which with the application of condition 1.3.2 yields \( B \subseteq \mathfrak{F} \), i.e., \( B = r(A) \), and \( r(A/r(A)) = 0 \). This proves condition 1.1.3.

Let us suppose that the conditions of Definition 1.1 are satisfied. The proof only requires condition 1.3.2. Let

\[
A \to B \to C \to 0
\]

be an exact sequence, and let \( A \subseteq \mathfrak{F} \), \( C \subseteq \mathfrak{F} \), whereupon \( \text{Im} \ x \subseteq \mathfrak{F} \), and \( r(B) \subseteq \text{Im} \ x \), so that \( B/r(B) \approx C/X \subseteq \mathfrak{F} \), and \( B = r(B) \).

**Remark 1.** Consider the functor \( F: \mathfrak{C} \to \mathfrak{D} \) from the category \( \mathfrak{C} \) of \( A \)-modules in some category with zero in which it is possible to define the concept of an exact sequence. Let \( F \) be exact on the right, and for any ascending sequence of modules \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots \); say that \( F(\bigcup A_\alpha) = 0 \) if \( F(A_\alpha) = 0 \) for all \( \alpha \). Definition 1.3 shows that the class of objects for which \( F(A) = 0 \) is radical. The converse can be proved by factorization of the category of modules with respect to the subcategory of radical modules.

**2. Hereditary Radicals**

**Proposition 2.1.** The following conditions are equivalent:

1) The functor \( r \) is exact on the left.

2) If \( A \subseteq B \), then \( r(A) \subseteq r(B) \subseteq A \)

3) If \( A \subseteq B \), and \( B \subseteq \mathfrak{F} \), then \( A \subseteq \mathfrak{F} \)

The proof is straightforward.

**Definition 2.1.** A radical \( r \) is called hereditary if it satisfies the conditions of Proposition 2.1.

If the functor is exact in Remark 1, then the radical defined by it is hereditary.

**Theorem 2.1.** If a ring \( A \) is hereditary, the functor \( R^0 r \) is the minimum hereditary radical containing \( r \).

**Proof.** The functor \( R^0 r \) is defined as follows:

\[
0 \to A \to Q \xrightarrow{\alpha} X \to 0
\]

is an exact sequence with injective module \( Q \); then

\[
R^0 r(A) = \text{Ker} \ r(\alpha) = r(Q) \cap A.
\]

We note that a radical \( r \) is hereditary if and only if \( R^0 r(A) = r(A) \) for any module \( A \). If \( A \) is injective, \( R^0 r(A) = r(A) \) for any radical \( r \).

The functor \( R^0 r \) is exact on the left and, clearly, satisfies the first two conditions of Definition 1.2. We prove the third condition. We assume in the beginning that \( A \) is injective, so that \( R^0 r(A/R^0 r(A)) = R^0 r(A/r(A)) \). Inasmuch as \( A \) is hereditary, \( A/r(A) \) is injective, and therefore

\[
R^0 r(A/r(A)) = r(A/r(A)) = 0.
\]