A criterion is given for finiteness of the number of nonisomorphic nondecomposable representations for the completely decomposable orders over a complete local Dedekind ring which comprise an intersection of maximal orders.

1. Preliminaries

In the article we use $\mathcal{O}$ to denote a complete local Dedekind ring with quotient field $k$. We let $\Lambda$ denote an $\mathcal{O}$-order in a separable $k$-algebra $\mathcal{A}$ [1]. A $\Lambda$-module $\Lambda$ is called a representation module if as an $\mathcal{O}$-module it is finitely generated and is without torsion. A submodule $B \subseteq \Lambda$ is called strict [2] if $\Lambda/B$ is a representation module. A representation module that does not contain nontrivial strict submodules is called irreducible. A module that decomposes into a direct sum of irreducible modules is called completely decomposable.

By a completely decomposable ring we mean an $\mathcal{O}$-order $\Lambda$ that is completely decomposable as a $\Lambda$-module. Clearly, a ring of this type decomposes into a direct sum of orders in simple $k$-algebras, hence we are permitted in the analysis of the representations to limit our discussion at once to the case when the algebra $\mathcal{A}$ is simple. Then all irreducible $\Lambda$-modules may be regarded as embedded in a unique simple $\Lambda$-module, forming a structure with respect to addition and intersection.

For any right (left) representation $\Lambda$-module $\Lambda$ it has defined for it a dual left (right) module $\Lambda^* = \text{Hom}_\mathcal{O}(\Lambda, \mathcal{O})$, where $\Lambda^{**} = \Lambda$, and the correspondence $\Lambda \rightarrow \Lambda^*$ determines the duality of the structures of right and left irreducible modules.

In the ensuing discussion we interpret an injective module to mean a module that is injective in the representation category [2] or, what amounts to the same thing, is dual to a projective module. If a module is simultaneously projective and injective (in the representation category), we call it bijective. We will have use for the following result, which is proved in [3].

Proposition 1.1. An irreducible module over a completely decomposable ring is projective (injective) if and only if it has one and only one maximal submodule (minimal supermodule).

We also recall the definition of the divisibility of modules [14]. We say that a module $\Lambda$ is divisible by a module $B$, writing $\Lambda \mid B$, if there exists an epimorphism $\Lambda^m \rightarrow B$, where $\Lambda^m$ is the direct sum of $m$ copies of $\Lambda$.

Throughout the article we assume that $\Lambda$ is a completely decomposable ring representing the intersection of maximal rings. In this case, if $B$ is a proper submodule of an irreducible module $\Lambda$, and $B \cong \Lambda$, then $B \subset AR$, where $R$ is the Jacobson radical of $\Lambda$. Moreover, all simple components of the module $\Lambda/AR$ are distinct, hence its length $l(\Lambda/AR)$ is simply the number of maximal submodules in $\Lambda$.

2. An Example

Let $D$ be a finite-dimensional separable field over $k$, $\Omega$ its unique maximal order [5], $\pi$ the generator of a maximal ideal $\pi$ of $\Omega$, and $M_n(D)$ the algebra of $n$th-order matrices over $D$. We consider the order $\Lambda_n$ in $M_n(D)$ ($n \geq 2$):
The radical $R$ of $\Lambda_n$ is equal to $\pi M_n(\Omega)$. Let $A$ be a representation $\Lambda_n$-module. Let us examine the exact sequence

$$0 \to AR \to A \to A/AR \to 0.$$ 

The ring of multipliers $AR$ is $M_n(\Omega)$, so that $AR = B^s$, where $B$ is the unique irreducible $M_n(\Omega)$-module. The module $A/AR$ is a $\Lambda_n/R$-module, hence $A/AR = B_1^{s_1} \oplus \ldots \oplus B_n^{s_n}$, where $B_1, \ldots, B_n$ are simple $\Lambda_n/R$-modules. Consequently, $A$ corresponds to an element $\alpha \in \text{Ext}^1_{\Lambda_n}(B_1^{s_1} \oplus \ldots \oplus B_n^{s_n}, B')$. It is readily seen that $\text{Ext}^1_{\Lambda_n}(B_1, B) \approx \Omega/\Pi = T$. Therefore $\alpha$ may be regarded as a matrix over $T$ of the form

$$\mathcal{M} = (\mathcal{M}_1 | \mathcal{M}_2 | \ldots | \mathcal{M}_n),$$

where $\mathcal{M}_i$ is an $s \times s_i$ matrix. The converse is easily verified, that every such matrix $\mathcal{M}$ corresponds to a representation $\Lambda_n$-module, provided only that

$$(2.1) \quad \text{rank } \mathcal{M} = s, \quad \text{rank } \mathcal{M}_i = s_i \quad (i = 1, \ldots, n).$$

Any automorphism of $A$ translates $AR$ into itself and therefore induces a transformation of $\mathcal{M}$. It is readily seen that every such transformation decomposes into a product of transformations of the following type:

- (2.2) elementary transformations of the rows of $\mathcal{M}$;
- (2.3) elementary transformations of the column of $\mathcal{M}_i \quad (i = 1, \ldots, n)$.

The module $A$ is decomposable if and only if the matrix $\mathcal{M}$ is decomposable by a sequence of transformations of the type (2.2) and (2.3). Thus, the problem of describing the representations of the ring $\Lambda_n$ is equivalent to that of reducing matrices $\mathcal{M}$ satisfying the conditions (2.1) by transformations of the type (2.2) and (2.3). We now show that this problem is equivalent to the description of representations of another ring.

Let $\Gamma_n$ be a subring of the direct sum $\Omega^n$ consisting of sets $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_n \mod \Pi$. The radical of $\Gamma_n$ is $\pi \Omega^n$, and $\Gamma_n/\pi \Omega^n = T$. Pursuing the same reasoning with regard to $\Gamma$ as for $\Lambda_n$, we see that the description of the representations of $\Gamma_n$ is tantamount to the reduction of matrices of the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1' \\ \vdots \\ \mathcal{M}_n' \end{pmatrix}$$

with coefficients from $T$, where any elementary transformations are allowed over the columns of $\mathcal{M}'$, the same being true over the rows inside every $\mathcal{M}_i'$, and if the dimensionality of $\mathcal{M}_i'$ is $s_i' \times s_i'$, then \( \text{rank } \mathcal{M}' = s' \), \( \text{rank } \mathcal{M}_i' = s_i' \).

Consequently, from any representation $\Lambda_n$-module we obtain a representation $\Gamma_n$-module by transposition of the corresponding matrix, and vice versa. Let us analyze the cases $n = 2, 3, 4$.

$n = 2$. It follows from [3] that every representation $\Lambda_2$-module is completely decomposable (as is easily verified by direct computation). There are altogether three nondecomposable (and irreducible) modules.

$n = 3$. The representations of $\Lambda_3$ (triplets) have been described by Bass [6].* There are eight nondecomposable representations. Therefore, $\Lambda_3$ also has eight nondecomposable representations, including seven irreducible members and the representation $C$ corresponding to the matrix

*See footnote on page following.