The author considers convex functionals \( f_t \) defined on the open convex subset \( G \) of real \( B \)-space \( X \) and continuous at \( x_0 \in G \). All linear functionals on \( X \) which are support functionals of form \( \int_T f_t(\omega) \mu(\omega) \) and \( \max_{t \in T} f_t(\omega) \) at \( x_0 \) are described.

Let \( G \) be a convex open subset of the real \( B \)-space \( X \) and let \( f \) be a convex functional defined on \( G \) and continuous at \( x_0 \in G \). The linear functional \( l \) on \( X \) is called a support functional to \( f \) at \( x_0 \) if \( l(x) - l(x_0) = f(x) - f(x_0) \) for all \( x \in G \). In addition, the set of functionals which are support functionals to \( f \) at point \( x_0 \) is called \( M(f, x_0) \); it is easy to see that \( M(f, x_0) \) is a convex weakly bicomplete subset \( X' \).

Support functionals are important in variation problems (see [1]).* In optimum control problems it is necessary to find a set \( M(f, x_0) \) when \( f \) is obtained with the aid of some operation from the functionals \( f_t \) for which the sets \( M(f_t, x_0) \) are known [1]. This problem also appears in extremum problems in approximation theory (see [2], [3]). The following relationships, in particular, were established in [1, 2]:

\[
M \left( \sum_{i=1}^{n} \alpha_i f_i, x_0 \right) = \sum_{i=1}^{n} \alpha_i M(f_i, x_0),
\]

where \( \alpha_i \) are positive numbers, \( t = 1, \ldots, n \), and

\[
M \left( \max_{t \in \{1, \ldots, n\}} f_t, x_0 \right) = \text{conv} \bigcup_{t \in \{1, \ldots, n\}} M(f_t, x_0),
\]

where conv denotes a convex shell and \( I_0(x_0) = \{t : f_t(x_0) = \max_{t \in \{1, \ldots, n\}} f_t(x_0)\} \).

The aim of this article is to extend these relationships to the case of an infinite set of functionals \( f_t \).

§1. Basic Results. We shall henceforth assume that \( G \) denotes an open convex subset in real \( B \)-space \( X, x_0 \in G \) and all considered functionals are assumed to be defined on \( G \), convex, and continuous at \( x_0 \).†

**Theorem 1.** Let \((T, \mu)\) be a space with measure ‡, \( t \in T \). We assume that the following conditions are satisfied: a) \( X \) is separable; b) \( f_t(x) \in \mathcal{L}^1(T, \mu) \) for any \( x \in G \); c) there exists a number \( \delta > 0 \) and a function \( c(t) \in \mathcal{L}^1(T, \mu) \), such that \( |f_t(x) - f_t(x_0)| \leq c(t) \) for all \( t \in T \) and all \( x, x_0 \in U_\delta(x_0) = \{x : \|x - x_0\| \leq \delta\} \).

Then the formula \( f(x) = \int_T f_t(x) \mu(\omega) \), \( x \in G \) defines the convex functional continuous at \( x_0 \) and \( M(f, x_0) \) consists of the functionals \( l \) which can be represented in the form \( l(x) = \int_T l_t(x) \mu(\omega) \), \( x \in X \), where \( l_t \in M(f_t, x_0) \) for any \( t \in T \) and \( l_t(x) \in \mathcal{L}^1(T, \mu) \) for any \( x \in X \).

If, moreover, \( c(t) \in \mathcal{L}^p(T, \mu) \), where \( 1 < p \leq \infty \), then \( l_t(x) \in \mathcal{L}^p(T, \mu) \) for any \( x \in X \). When the set \( T \) is countable we do not require \( X \) to be separable.

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*G. Minkovskii also considered support functionals in the finite dimensional case in association with problems on the geometry of numbers; since that time they have been widely used in various problems.
†Continuity on \( G \) actually follows from continuity at \( x_0 \) and convexity on the open set \( G \) (see [4]); however, this will not be required.
‡This means that the \( \sigma \)-algebra \( \Sigma \) of subsets \( T \) and \( \mu \) (the non-negative countably additive function on \( \Sigma \) ) is specified. For integration over the measure \( \mu \) and spaces \( \mathcal{L}^p(T, \mu) \) see [5].
E. G. Gold'shtein [3] previously obtained a similar result using much stronger assumptions.*

**Corollary.** Let \( 0 < \alpha_t \leq C < \infty, t = 1, 2, \ldots \) If \( \{f_t(x)\} \in \ell^1 \) for any \( x \in G \) and there exist \( c = (c_t) \in \ell^1 \) and \( \delta > 0 \), such that \( |f_t(x) - f_t(x_0)| \leq c_t \) for all \( x \in U_\delta(x_0) \) and all \( t = 1, 2, \ldots \), then \( \sum_{t=1}^{\infty} a_t(x) \) is a convex functional on \( G \) which is continuous at \( x_0 \) and \( M \left( \sum_{t=1}^{\infty} a_t(x), x_0 \right) \) consists of functionals \( l \) which can be represented in the form \( l(x) = \sum_{t=1}^{\infty} a_t(x), x \in X \), where \( l_t \in M(f_t, x_0), t = 1, 2, \ldots \), and \( \{l_t(x)\} \in \ell^1 \) for any \( x \in X \).

**Theorem 2.** Let \( T \) be a set \( \sup_{t \in T} f_t(x) < \infty \) for any \( x \in G \) and let there exist \( c > 0, \delta > 0 \), such that \( |f_t(x) - f_t(x_0)| \leq c \) for all \( t \in T \) and all \( x \in U_\delta(x_0) \). Then \( \sup_{t \in T} f_t(x) \) is a convex functional on \( G \) which is continuous at \( x_0 \) and

\[
M(\sup_{t \in T} f_t, x_0) = \left( \bigcap_{\epsilon > 0} \text{conv} \bigcup_{t \in T} M(f_t, x_0) \right),
\]

where the bar denotes weak closure and \( J_{\epsilon}(x_0) = \{t \in T : f_t(x_0) \geq \sup_{t \in T} f_t(x_0) - \epsilon\} \).

**Theorem 3.** Let \( T \) be a bicompactum. We assume that the following conditions are satisfied: a) \( X \) is separable; b) the function \( f_t(x) \) is continuous on \( T \) for all \( x \in G \); c) there exist \( c > 0, \delta > 0 \), such that \( |f_t(x) - f_t(x_0)| \leq c \) for all \( t \in T \) and all \( x \in U_\delta(x_0) \). Then \( f(x) = \max_{t \in T} f_t(x) \) is a convex functional on \( G \) which is continuous at \( x_0 \) and \( M(f, x_0) \) consists of the functionals \( l \) which can be represented as \( l(x) = \int_T f_t(x)p(dt) \), where \( p \) is the regular Borel measure on \( T, \mu(T) = 1 \), the carrier \( \mu \) lies in \( L_0(x_0) = \{t \in T : f_t(x_0) = f(x_0)\} \), and \( l_t \in M(f_t, x_0) \) for any \( t \in T \) and \( l_t(x) \in L^\infty(T, \mu) \) for any \( x \in X \).

If the set \( T \) is countable we do not require \( X \) to be separable.

§2. Proof. Let \( f \) be a convex functional defined on \( G \) and continuous at \( x_0 \). Since \( f \) is convex it follows that for any \( h \in X \) the ratio \( \frac{f(x_0 + \epsilon h) - f(x_0)}{\epsilon} \) approaches some finite limit \( \epsilon \downarrow 0 \) when \( dfx_0(h) \). It is not difficult to show that \( dfx_0 \) is a semiadditive positive uniform functional on \( X \) (see, for example, [1]).

**Lemma 1.** The functional \( dfx_0 \) is continuous on \( X \) and \( M(f', x_0) = M(dfx_0, 0) \).

We omit the proof in view of its triviality.

Let for any \( t \in T \) the semiadditive positive uniform functional \( p_t \) be specified on \( X \), where \( p_t(x) \in L^1(T, \mu) \) for any \( x \in X \) and \( p_t(x) \leq c(t) \|x\| \) for all \( t \in T, x \in X \), where \( c(t) \in L^1(T, \mu) \). Consider the mapping \( F : X \rightarrow L^1(T, \mu), F(x) = p_t(x), x \in X \); it immediately follows from the properties of \( p_t \) that \( F \) is continuous.

Let \( l_t, t \in T \) be a family of linear functionals on \( X \) satisfying the following conditions: 1) \( l_t \in M(p_t, 0) \) for any \( t \in T \); 2) \( l_t(x) \in L^1(T, \mu) \) for any \( x \in X \). We associate the linear mapping \( A : X \rightarrow L^1(T, \mu) \), which operates according to the formula \( A(x)(t) = l_t(x), x \in X \) with each such family.

**Lemma 2.** For any \( Ax \equiv F(x) \), we have \( x \in X \). If \( X \) is separable, then any linear mapping \( A \) possessing this property produces some family of functionals \( l_t, t \in T \) satisfying conditions 1) and 2).

**Proof.** It is clear that only the second statement is required in the proof. Let \( \Gamma \) be the Hamel basis (maximum linearly independent system of elements) in \( L^1(T, \mu) \) [6]. Then, as is known, any element in \( L^1(T, \mu) \) is represented uniquely in the form of a finite combination of elements \( \Gamma \). The classes of functions

*The author of [3] considered the topological space \( T \) with Borel measure \( \mu, \mu(T) < \infty \) and assumed that: a) \( X \) is separable; b) \( f_t(x) \) (as a function of \( t \)) is continuous and bounded on \( T \) for any \( x \in G \); c) \( f_t(t) \) as a function of two variables \( (t, x) \) is uniformly continuous on \( T \times U_\delta(x_0) \) for some \( \delta > 0 \). Under these assumptions, a representation of any \( l \in M(f_t, x_0) \) in the form \( l(x) = \int_T l_t(x)\mu(dt) \) was obtained in [3], where \( l_t \in M(f_t, x_0) \) and \( l_t(x) \) are \( \mu \)-measurable for any \( x \in X \).

†I.e., \( dfx_0(h_1 + h_2) \leq dfx_0(h_1) + dfx_0(h_2) \) for any \( h_1, h_2 \in X \) (semiadditivity) and \( dfx_0(\alpha h) = \alpha dfx_0(h) \) for any \( h \in X, \alpha \geq 0 \) (positive uniformity).