ON THE MUTUAL GROWTH OF NEIGHBORING COEFFICIENTS OF UNIVALENT FUNCTIONS

L. P. Il'ina

In the class $S$ of functions $f(z) = z + c_2z^2 + c_3z^3 + \ldots$, regular and univalent in $|z| < 1$, the following bound is obtained: $|c_{n+1}|/|c_n| < 4.26$, $n = 1, 2, \ldots$

In this paper we consider the following two classes of functions:

$S$, the class of functions $f(t) = t + \sum_{n=2}^{\infty} c_n t^n$, regular and univalent in the disk $|t| < 1$;

$\Sigma$, the class of functions $F(z) = z + \alpha_0 + \alpha_1z^{-1} + \ldots$, meromorphic and univalent in the domain $|z| > 1$.

The order of growth of the coefficients for the class $S$ was obtained by Littlewood [1] in 1925:

$$|c_n| < a_n, \quad n = 2, 3, \ldots$$

In 1964, I. M. Milin [2] proved that

$$|c_n| < 1.243n, \quad n = 2, 3, \ldots$$

With respect to the growth of neighboring coefficients, a smaller order of growth was obtained in 1946 by G. M. Goluzin (see [3], pp. 193-195):

$$||c_{n+1}| - |c_n|| < C n^{\alpha} \ln n, \quad n = 2, 3, \ldots,$$

where $C$ is an absolute constant. This bound was later improved by M. Biernacki [4]:

$$||c_{n+1}| - |c_n|| < C (\ln n)^{\alpha},$$

subsequently, Kheinman [5] obtained the following result:

$$||c_{n+1}| - |c_n|| < C, \quad n = 1, 2, \ldots$$

Only recently, I. M. Milin [6] showed that in this inequality $C < 9$.

We shall show below that $C < 4.26$.

**THEOREM.** The following inequality holds for each $n = 1, 2, \ldots$:

$$f(t) = t + c_2 t^2 + \ldots + c_n t^n + \ldots$$

for the functions

$$||c_{n+1}| - |c_n|| < 4.26. \quad (1)$$

**Proof.** Let us take an arbitrary function $f(t) \in S$, and let us form from it the function $F(z) \in \Sigma : F(z) = [f(z^{-1})]^{-1}, |z| > 1$. Let us now select a value of $t$ on the circle $|z| = \rho > 1$, such that

$$|F(t)| = \min_{|z|=\rho} |F(z)|, \quad (2)$$

and let us form the function $(1-t/z)f'(1/z)$. According to the Cauchy formula for the Taylor coefficients of this function, we shall have

$$(n + 1)c_{n+1} - nc_n = \frac{1}{2\pi i} \sum_{|z|=\rho} \left(1 - \frac{t}{z}\right)f'(\frac{1}{z}) z^n dz.$$ \quad (3)
Noting that
\[ f'(1/z) = z^2 F'(z) / F^2(z), \]
we may rewrite Eq. (3) thus:
\[ \begin{align*}
(n + 1) c_{n+1} - n c_n t &= \frac{1}{2\pi i} \int_{|z|=\rho} \frac{z z F'(z)}{F(z) - F(t)} \frac{z - t}{z - \overline{t}} \\
&\times \left( \frac{1}{F'(z)} - \frac{1}{z} \right) \frac{z - t}{z - \overline{t}} \left( 1 - \frac{F(t)}{F(z)} \right) \frac{z - t}{z - \overline{t}} z^n \frac{dz}{z}.
\end{align*} \]
(4)

Each function \( F(z) \in \Sigma \) generates, through the expansion
\[ \ln \frac{z - t}{F(z) - F(t)} = \sum_{k=1}^{\infty} A_k(t) z^{-k}, \quad |z| > 1, \quad |t| > 1, \]
a system of functions \( \{A_k(z)\}_1^{\infty} \). From the properties of the system \( \{A_k(z)\}_1^{\infty} \) (see [7]), we note the inequality
\[ \sum_{k=1}^{\infty} k |A_k(z)|^2 < \ln \rho, \quad |z| = \rho > 1. \]
(6)

Further, for an arbitrary sequence of complex numbers \( \{A_k\}_1^{\infty} \), which generates a sequence \( \{D_k\}_1^{\infty} \) by means of the expansion
\[ \exp \sum_{k=1}^{\infty} A_k z^k = \sum_{k=1}^{\infty} D_k z^k, \]
we can obtain, for \( n = 1, 2, \ldots, \) the following three inequalities (see [8, 9]):
\[ |D_n| \leq \exp \left\{ \frac{1}{2} \left( \sum_{k=1}^{n} k |A_k|^2 - \sum_{k=1}^{n} \frac{1}{k^2} \right) \right\}, \]
(7)
\[ \sum_{k=1}^{n} |D_k|^2 \leq (n + 1) \exp \left\{ \sum_{k=1}^{n+1} k |A_k|^2 - \sum_{k=1}^{n+1} \frac{1}{k^2} + 1 - \frac{1}{n + 1} \sum_{k=1}^{n+1} k^2 |A_k|^2 \right\}, \]
(8)
\[ \sum_{k=1}^{\infty} |D_k|^2 \leq \exp \left( \sum_{k=1}^{\infty} k |A_k|^2 \right). \]
(9)

Let us now return to Eq. (4). Noting that
\[ z \left( \ln \frac{z - t}{F(z) - F(t)} \right)' = \frac{z}{z - t} - \frac{z F'(z)}{F(z) - F(t)}, \]
and
\[ z \left( \frac{z - t}{F(z) - F(t)} \right)' = z \left( \ln \frac{z}{F(z) - F(t)} \right)' = \frac{z - t}{z - \overline{t}}, \]
we can transform the first term of the integrand in Eq. (4) in the following way:
\[ \begin{align*}
\frac{z F'(z)}{F(z) - F(t)} &\frac{z - t}{z - \overline{t}} \left( 1 - \frac{F(t)}{F(z)} \right) = -Z' \left[ \frac{z - t}{F(z) - F(t)} \right] \\
&= -\left[ \frac{z F'(z)}{F(z) - F(t)} - \frac{z}{z - \overline{t}} \right] \left( \frac{z - t}{z - \overline{t}} \right) \frac{F(t)}{F(z)} + \frac{z}{z - \overline{t}} \frac{f' \left( \frac{z - t}{z} \right) F(t)}{F(z)}.
\end{align*} \]
(11)

If we take this into account, and if we also use the Taylor series expansions for \( |z| > 1 \) of the functions \( (z - t)/(F(z) - F(t)) \) and \( z/F(z) \), namely
\[ \frac{z - t}{F(z) - F(t)} = \sum_{k=1}^{\infty} D_k(t) z^{-k}, \quad D_0(t) = 1, \]
\[ \frac{z}{F(z)} = \sum_{k=1}^{\infty} c_{k+1} z^{-k}, \quad c_1 = 1, \]
we can rewrite formula (4) as follows:
\[ \begin{align*}
(n + 1) c_{n+1} - n c_n t &= n D_n(t) - \frac{1}{2\pi i} \sum_{|z|=\rho} \left[ \left( \frac{z F'(z)}{F(z) - F(t)} \right) - \frac{z}{z - \overline{t}} \right] \frac{F(t)}{F(z) - F(t)} + \frac{1}{z} \frac{f' \left( \frac{z - t}{z} \right) F(t)}{F(z) - F(t)} z^n \frac{dz}{z}.
\end{align*} \]
(13)

*By using a representation of this kind for the integrand, I. M. Milin obtained the result mentioned above.