Total Algebras and Weak Independence. II

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1. Introduction

The theory of primal algebras has recently been generalized in several ways. In particular the concept has been extended on the one hand by the admission of subalgebras and internal isomorphisms to the successively larger classes of semi-primal, infra-primal and quasi-primal algebras [1, 3, 8], and on the other hand by the admission of proper congruences to the hemi-primal algebras [2]. In [5] the authors introduced the concept of “total” algebras: finite algebras in which all mappings which preserve all subalgebras, automorphisms and congruences are representable by polynomials. So defined, this class of algebras was shown to comprehend both the infra-primal and hemi-primal algebras, but, on the other hand, it is easy to see that the quasi-primal algebras (since they may possess non-identical isomorphic subalgebras) are not always total.

In the present paper we enlarge the class of total algebras to include the quasi-primals as well as the semi-, infra-, and hemi-primals. Our principal focus is on systems of weakly independent quasi-primal algebras, their products, and the varieties generated by them.

2. Notation; Weak Independence

All algebras are assumed, unless otherwise stated, to be of the same fixed type $\tau$. As in [5] $\mathcal{A}$ will denote the (equational) theory of algebra $\mathcal{A}$. If $\mathcal{E}$ is any equational theory of type $\tau$, $L(\mathcal{E})$ denotes the sublattice of theories $\succeq \mathcal{E}$ in the lattice of all theories of type $\tau$, ordered by inclusion. $\mathcal{E}_1 \vee \mathcal{E}_2$ and $\mathcal{E}_1 \wedge \mathcal{E}_2$ denote the join and meet respectively of $\mathcal{E}_1$ and $\mathcal{E}_2$ and $1$ denotes the unit element of $L(\mathcal{E})$. Note that for algebras $\mathcal{A}, \mathcal{B}$, $|\mathcal{A}| \wedge |\mathcal{B}| = |\mathcal{A} \times \mathcal{B}|$. We recall that there is a 1-1 correspondence between varieties (equational classes) of algebras and equational theories and that, for a variety $\mathcal{V}$ and corresponding theory $\mathcal{E}$, $L(\mathcal{E})$ is dually isomorphic with the lattice of subvarieties of $\mathcal{V}$.

Let $G$ be a set and $\mathcal{E}$ an equational theory. By $\mathfrak{I}(G, \mathcal{E})$ we denote the $\mathcal{E}$-free polynomial algebra with free generating set $G$. The elements of $\mathfrak{I}$ are (equivalence classes of) polynomial symbols over the elements.
of \( G \); two polynomial symbols \( \Phi_1, \Phi_2 \) are considered equal provided \( \Phi_1 = \Phi_2 \in \mathcal{E} \).

If \( \mathcal{A} \) is an algebra, \( \Theta(\mathcal{A}) \) denotes the lattice of congruence relations of \( \mathcal{A} \).

Equational theories, \( \mathcal{E}_1, \ldots, \mathcal{E}_n \), are said to be weakly independent [5] if for each set \( \Phi_1, \ldots, \Phi_n \) of polynomial symbols for which \( \Phi_i = \Phi_j \in \mathcal{E}_i \vee \mathcal{E}_j \) \((i, j = 1, \ldots, n)\) there is a polynomial symbol \( \Phi \) for which \( \Phi = \Phi_i \in \mathcal{E}_i \) \((i = 1, \ldots, n)\). If the theories of a finite set of algebras are weakly independent we say that the set of algebras is weakly independent. A family of algebras such that each finite subset is independent is called a weak cluster of algebras.

An algebra \( \mathcal{B} \) is a co-subalgebra of algebras \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) if \( \mathcal{B} \) is isomorphic with subalgebras of each of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). If \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are algebras and for each set \( \Phi_1, \ldots, \Phi_n \) of polynomial symbols such that \( \Phi_i = \Phi_j \) \((i, j = 1, \ldots, n)\) is an identity of each co-subalgebra of \( \mathcal{A}_1 \) and \( \mathcal{A}_j \), there is a polynomial symbol \( \Phi \) such that \( \Phi = \Phi_i \in [\mathcal{A}_1], i = 1, \ldots, n \), then \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are said to be subweakly independent. Subweak clusters are defined analogously to weak clusters. Evidently subweak independence implies weak independence.

A consequence of the following Lemma is, for example, that any pair of groups or of rings is weakly independent.

**Lemma 1.** Let \( \mathcal{V} \) be a variety of algebras such that \( \Theta(\mathcal{A}) \) is permutable for all \( \mathcal{A} \in \mathcal{V} \). Then every pair of algebras in \( \mathcal{V} \) is weakly independent.

**Proof.** Let \( G \) be a countable set. Let \( \mathfrak{A} = \mathfrak{A}(G, \mathcal{E}) \) where \( \mathcal{E} \) is the theory corresponding to \( \mathcal{V} \). For \( \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{V} \), let \( \mathfrak{A}_i = \mathfrak{A}(G, [\mathcal{A}_i]), i = 1, 2 \), and \( \mathfrak{A}_{12} = \mathfrak{A}(G, [\mathcal{A}_1] \vee [\mathcal{A}_2]) \). Then for certain congruences \( \theta_1, \theta_2 \) on \( \mathfrak{A}, \mathfrak{A}_1 \cong \mathfrak{A}_i \), so that each \( \mathfrak{A}_i \) is homomorphic to \( \mathfrak{A} / (\theta_1 \vee \theta_2) \) and hence \( \mathfrak{A}_{12} \) is isomorphic to \( \mathfrak{A} / (\theta_1 \vee \theta_2) \). Suppose \( \Phi_1 = \Phi_2 \in [\mathcal{A}_1] \vee [\mathcal{A}_2] \). Then \( \Phi_1 = \Phi_2 \) in \( \mathfrak{A}_{12} \) and hence in \( \mathfrak{A} / (\theta_1 \vee \theta_2) \). But since \( \mathfrak{A} \in \mathcal{V} \), there is a polynomial symbol \( \Phi \) such that \( \Phi_1 \theta_1 \Phi \) and \( \Phi_2 \theta_2 \Phi \). Thus \( \Phi = \Phi_1 \Phi_2 \) in \( \mathfrak{A}_i \), so that \( \Phi = \Phi \in [\mathcal{A}_1], i = 1, 2 \).

**Theorem 2.2.** Let \( \mathcal{V} \) be a variety of algebras such that \( \Theta(\mathcal{A}) \) is permutable and distributive for every \( \mathcal{A} \in \mathcal{V} \). Then every finite subset of \( \mathcal{V} \) consists of weakly independent algebras, i.e., \( \mathcal{V} \) is a weak cluster.

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1. i.e.: every pair of elements of \( \Theta(\mathcal{A}) \) permute with respect to relation product. According to [7] this is equivalent to the existence of a ternary polynomial \( \Phi \) for which \( \Phi(x, y, x) = x \in \mathcal{E} \), the equational theory corresponding to \( \mathcal{V} \).

2. According to [9] this is equivalent to the existence of a ternary polynomial symbol \( \Phi \) for which

\[
\Phi(x, y, y) = \Phi(y, y, x) = \Phi(x, y, x) = x \in \mathcal{E},
\]

the equational theory corresponding to \( \mathcal{V} \).