A GENERALIZATION OF A THEOREM OF M. RIESZ TO THE CASE OF FUNCTIONS OF SEVERAL VARIABLES

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The following theorem was proved by M. Riesz: If \( f(x) \in L(-\pi, \pi), f(x) \geq 0 \) and the conjugate function \( \tilde{f}(x) \) is also integrable on \([-\pi, \pi]\), then \( f(x) \in L\log^+L \). The analog of this theorem for functions of several variables is established.

§ 1. Introduction

Assume that the \( 2\pi \)-periodic function \( f(x) \) is integrable on \([-\pi, \pi]\). It is known ([1], p. 528) that the conjugate function

\[ \tilde{f}(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + t) \cotg \frac{t}{2} \, dt \]

exists almost everywhere.

The following theorem is well known.

\textbf{Theorem (A. Zygmund).} If \( f(x) \in L \log^+L \) in \([-\pi, \pi]\), then \( \tilde{f}(x) \in L(-\pi, \pi) \) and

\[ \int_{-\pi}^{\pi} |\tilde{f}(x)| \, dx \leq A \int_{-\pi}^{\pi} |f(x)| \log^+ |f(x)| \, dx + B \]

where \( A \) and \( B \) are positive absolute constants.

If \( f(x) \geq 0 \), Zygmund's theorem has the following converse ([3] and [1], p. 571).

\textbf{Theorem (M. Riesz).} If \( f(x) \geq 0 \) and \( f(x) \) and \( \tilde{f}(x) \) are both integrable over \([-\pi, \pi]\), then \( f(x) \in L \log^+L \) in \([-\pi, \pi]\).

We note that more general results ([4], p. 468) in the direction of Zygmund's and Riesz's theorems have been established.

In the present article we consider an analog of Riesz's theorem for functions of several variables.

§ 2. The Conjugate Function of Two Variables

Let \( f(x, y) \) be integrable on \( R = [-\pi, \pi; -\pi, \pi] \). We assume that \( f(x, y) \) is periodic with period \( 2\pi \) in each of the variables.

Consider the following conjugate functions of two variables:

\[ \tilde{f}_1(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + s, y) \cotg \frac{s}{2} \, ds, \quad \tilde{f}_2(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y + t) \cotg \frac{t}{2} \, dt, \]

\[ \tilde{f}_3(x, y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) \cotg \frac{s}{2} \cotg \frac{t}{2} \, ds \, dt. \]

A theorem due to Lusin and Privalov ([1], p 528) states that the functions \( \tilde{f}_i(x, y) \) (\( i = 1, 2 \)) exist almost everywhere for every function \( f(x, y) \in L(R) \). Zygmund [5] proved that if \( f(x, y) \in L\log^+L \) in \( R \) then \( \tilde{f}_3(x, y) \) exists almost everywhere. The condition \( f(x, y) \in L(\log^+L)^{1-\varepsilon} \) in \( R \) for any \( \varepsilon \in (0, 1) \) does not in general guarantee the existence of \( \tilde{f}_3(x, y) \) on a set of positive plane measure [6, 7, 8].
Moreover if we use (1) we have
\[ \sum_{n=1}^{\infty} f(x, y) \, dx \, dy \leq A \sum_{n=1}^{\infty} f(x, y) \, \log^+ f(x, y) \, dx \, dy + B \quad (i = 1, 2), \]
i.e., the \( f_i(x, y) \) (i = 1, 2) are integrable in \( R \) if \( f(x, y) \in \text{Log}^+ \).

As to \( f_3(x, y) \), it is integrable on \( R \) if \( f(x, y) \in \text{Log}^+ \) \( \epsilon \) \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y) \, dx \]
i.e., if \( f(x, y) \geq 0 \), (x, y) and \( f_1(x, y) \) are both integrable on \( R \) then \( f(x, y) \in \text{Log}^+ \) (and the same result holds if \( f_1 \) is replaced by \( f_2 \)). But a simple example shows that this is not generally true.

It would appear that the analog of Riesz’s theorem holds trivially for the functions \( f_i(x, y) \) (i = 1, 2), i.e., if \( f(x, y) \rightarrow 0 \), \( f(x, y) \) and \( f_1(x, y) \) are both integrable on \( R \), then \( f(x, y) \in \text{Log}^+ \) (and the same result holds if \( f_1 \) is replaced by \( f_2 \)). But a simple example shows that the answer is negative. Hence, it is natural to pose the question: what is the analog of Riesz’s theorem for functions of two variables?

We obtain results which supply the answer to this question.

**Theorem 1.**

a) Let \( f(x, y) \geq 0 \). If \( f(x, y) \) and \( \| f_1(x, y) \| \text{Log}^+ f_1(x, y) \| (i = 1, 2) \) are integrable in \( R \), then \( f(x, y) \in \text{Log}^+ \).

b) There exists a nonnegative \( 2\pi \)-periodic function \( f(x, y) \) such that \( f(x, y) \in \text{Log}^+ \), \( f_1(x, y) \in \text{Log}^+ \) and \( f_2(x, y) \in \text{Log}^+ \) on \( R \) for all \( \gamma \in [0, 1) \), but \( f(x, y) \not\in \text{Log}^+ \).

**Proof.** Let
\[ \phi(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y) \, dx. \]
Plainly \( \phi(y) \) is integrable on \([\pi, \pi] \), \( 2\pi \)-periodic, and \( \phi(y) \geq 0 \). It is easily verified that
\[ \phi(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(x, y) \, dx \]
for almost all \( y \). Hence \( \phi(y) \in \text{L}(-\pi, \pi) \) since \( f_2(x, y) \in \text{L}(R) \). It follows from Riesz’s theorem that
\[ \int_{-\pi}^{\pi} \phi(y) \, dy \, d\gamma = O(1). \]
Now let
\[ u \equiv u(y, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s, y) \, P_r(s - x) \, ds, \]
\[ v \equiv v(y, z) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s, y) \, Q_r(s - x) \, ds; \quad z = re^{ix}, \quad 0 < r < 1, \]
where
\[ P_r(x) = \frac{1}{2\pi} \left( 1 - r^2 \cos^2 \alpha \right)^{1/2}, \quad Q_r(x) = \frac{r \sin x}{1 - 2r \cos \alpha + r^2}. \]
With no loss of generality we can assume that \( f(x, y) \geq 2 \) [otherwise we can consider the function \( g(x, y) = f(x, y) + 2 \)]; consider the function \( F(y, z) = u(y, z) + iv(y, z) \equiv u + iv \) analytic in the disk \( |z| < 1 \) (for almost all fixed \( y \)), where \( u \equiv 2 \). Cauchy’s formula yields
\[ \frac{1}{2\pi} \int_{|z|=1} \frac{F(y, z) \, dz}{z} = P(y, 0) \log F(y, 0). \]
Hence by equating real parts and using (2) we have
\[ \text{Re} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [u + iv] \log \sqrt{u^2 + v^2 + \theta^2} \, dx \right\} \sim \frac{1}{2\pi} \int_{-\pi}^{\pi} u \log u \, dx = \phi(y) \log \phi(y), \]
where \( \theta = \text{arg} F(y, z) \). But since \( |\theta| \leq \pi/2 \), the last relation implies
\[ \frac{1}{4} \int_{-\pi}^{\pi} u \log u \, dx \leq \frac{1}{4} \int_{-\pi}^{\pi} |v| \, dx + \phi(y) \log \phi(y). \]

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