A theorem on the existence of isomorphic continuations for free decompositions of a $\Gamma$-operator group with a group of operators $\Gamma$ acting regularly is proved.

This paper is devoted to proving a theorem on the existence of isomorphic continuations for free decompositions of a $\Gamma$-operator group $G$ with a regular group of operators $\Gamma$. This latter means that $g\alpha = g$ if and only if $g = 1$, or $\alpha = e$ is the unit of the group $\Gamma$.

Let us recall that a free product $G = \ast_{i \in I} A_i$ of $\Gamma$-operator groups $A_i$, $i \in I$ is constructed as follows:

$$
G = \ast_{i \in I} A_i,
$$

where if $g = a_1 \ldots a_n \in G$ and $\alpha \in \Gamma$, then $g\alpha = (a_1\alpha) \ldots (a_n\alpha)$. An isomorphism of two free decompositions $G = \ast_{i \in I} A_i = \ast_{j \in J} B_j$ of a $\Gamma$-operator group $G$ will be understood in the following sense: a mutually one-to-one correspondence can be established between the free factors of both decompositions such that corresponding factors of $A_i$ and $B_j$ will be isomorphic, where if $A_i$ and $B_j$ are not free $\Gamma$-groups, there then exist also conjugate subgroups $A_i^\perp \subseteq A_i$, $B_j^\perp \subseteq B_j$ in $G$ which generate admissible subgroups of $A_i$ and $B_j$ such that the isomorphism between $A_i$ and $B_j$ is a continuation of the conjugacy between $A_i^\perp$ and $B_j^\perp$.

To prove Theorem 2 on isomorphic continuations, some admissible subgroups of the free product of $\Gamma$-operator groups are first described in Theorem 1. The proof of this theorem is a simple transferral of the proof given by MacLane [1], of the A. G. Kurosh theorem on subgroups of a free product, and is hence not presented herein. We limit ourselves to a remark on the changes which must be inserted into this proof because of the presence of the operators, and we refer the reader to Chapter 3, sec. 8, in the book [2].

Before turning to Theorem 1, let us make some remarks. Let us consider an admissible subgroup $H$ in $G = \ast_{i \in I} A_i$ such that for any $g \in G$ and $\alpha \in \Gamma$, $\alpha \neq e$, there follows $g \in H$ from $g(\alpha^{-1}) \in H$. In particular, free factors of a $\Gamma$-operator group $G$ with a regular group of operators $\Gamma$, which are of special interest to us, possess this property.

The constraint imposed on the subgroup $H$ is equivalent to the following condition: for any adjacent class $C = xH \neq H$ and $\alpha \in \Gamma$, $\alpha \neq e$, there will be $C\alpha = (x\alpha)H \neq C$. Hence, if we select the element $x$ as representative of the class $xH$, the element $x\alpha$ can then be taken as representative of the class $(x\alpha)H$.

Let us assume that $A, H$ are admissible subgroups of the $\Gamma$-operator group $G$. Let us consider the decomposition of the group $G$ in a double modulus $G = \bigcup (AgH)$. If $D = AgH$ is the class of this decomposition, then by putting

$$
D_{\Gamma} = (AgH)_{\Gamma} = \bigcup_{g \in G} (A(g\alpha)H),
$$

we obtain the decomposition $(A, H)$ of the group $G$ in the nonintersecting class $D_{\Gamma}$, $G = \bigcup_{g \in G} (AgH)_{\Gamma}$. Let us note that the class $(AH)_{\Gamma}$ coincides with the class $AH$.

**Theorem 1.** Let a group $G$ be a free product of $\Gamma$-groups $A_i$, $i \in I$, and let $H$ be an admissible subgroup of the group $G$ which satisfies the condition: for any $g \in G$, $\alpha \in \Gamma$, $\alpha \neq e$, there follows $g \in H$ from...
g(gα⁻¹) ∈ H. Then in each class DΓ a representative s = s(i, DΓ) can be selected from the decomposition 
(Ai, H, i ∈ I, such that

\[ H \simeq F \ast \bigoplus_{i,s} (H \cap s^{-1}A_i)s \Gamma, \]

where \((H \cap s^{-1}A_i)s \Gamma\) is a \(\Gamma\) admissible subgroup generated by the subgroup \(H \cap s^{-1}A_i, F\) is a \(\Gamma\)-free group. The representative is selected in such a manner that the representative of the class \(A_iH\) is 1 for all \(i \in I\).

A proof of this theorem for the operator-free case is presented in Chapter 3, §8 in [2]. Only the following changes are inserted in the presence of a group of operators \(\Gamma\). In formulating Lemma 1 it must be demanded that the condition \(3)' \tau_i(C\alpha) = \tau_i(C)\alpha\) be satisfied for all \(\alpha \in \Gamma\). To the identities a)-b) preceding Lemma 3 must be added the identity \(\Gamma(C, i, j) = [C, i, j]\alpha, and in Lemma 3 itself it must be shown that a \(\Gamma\)-operator group \(F\) with relations a)-\(\Gamma\) is \(\Gamma\)-free. Moreover, in place of the intersection \(H \cap s^{-1}A_i\) it is necessary to consider the group \((H \cap s^{-1}A_i)s \Gamma \) throughout.

By utilizing Theorem 1 the theorem on isomorphic continuations can be proved. Let us assume that a group of operators \(\Gamma\) acts regularly on a group \(G\), i.e., \(g\alpha = g\) if and only if either \(g = 1\), or \(\alpha = e\). In this case, as has already been remarked above, each free factor of the group \(G\) satisfies the condition imposed on the subgroup \(N\) in Theorem 1.

**THEOREM 2.** Two arbitrary free decompositions of a \(\Gamma\)-operator group \(G\) with a regular group of operators \(\Gamma\) possess isomorphic continuations.

**Proof.** Let the group \(G\) possess two decompositions

\[ G = \bigoplus_{i \in I} A_i, \]  
(1)

\[ G = \bigoplus_{j \in J} B_j, \]  
(2)

Because of Theorem 1 applied to each \(A_i\) and the decomposition (2), we will have

\[ A_i \simeq F_i \ast \bigoplus_{(A_i \cap s^{-1}B_j)s_{ij}} \],

where the elements \(s_{ij}\) generate a complete system of representatives of the classes \((B_j \cap s^{-1}A_i) \Gamma\). Analogously for each \(B_j\) and the decomposition (1) we will have

\[ B_j \simeq F_j \ast \bigoplus_{(B_j \cap s^{-1}A_i)t_{ij} \Gamma} \],

where \(t_{ij}\) is a complete system of representatives of the classes \((A_i \cap s^{-1}B_j) \Gamma\). Therefore, in conformity with (3) and (4) we can continue the decompositions (1) and (2). Let us show that the decompositions of the group \(G\) which have been obtained are isomorphic.

Let us assume that a factor \((B_j \cap s^{-1}A_i) \Gamma\) enters into the product (4), where \(t\) is a representative of the class \((A_i \cap s^{-1}B_j) \Gamma\). Hence, an arbitrary element \(g \in (B_j \cap s^{-1}A_i) \Gamma\) has the length \(l(g) = 2l(t) + 1\) relative to the decomposition (1).

If it is assumed that \(t = 1\), then \((B_j \cap s^{-1}A_i) \Gamma = B_jA_i\), and by virtue of Theorem 1 \(s = s(j, B_jA_i) = 1\). Hence, \((B_j \cap A_i) \Gamma = B_j \cap A_i = (A_i \cap B_j) \Gamma\).

Now, let \(t \neq 1\), and therefore, \(s \neq 1\). Let us show that in this case the decompositions

\[ (B_j \cap s^{-1}A_i) \Gamma = \bigoplus_{r \in \mathbb{T}} [B_j \cap (t\gamma)^{-1}A_i(t\gamma)] \]  
(5)

\[ (A_i \cap s^{-1}B_j) \Gamma = \bigoplus_{r \in \mathbb{T}} [A_i \cap (t\gamma)^{-1}B_j(t\gamma)] \]  
(6)

hold in the category of all operator-free groups. Let us consider the former, say. It follows from the definition of \(t\) (see the end of the proof of Lemma 1 to Theorem 1) that its first subscript is different from \(i\), hence, an arbitrary element \(g \in B_j \cap (t\gamma)^{-1}A_i(t\gamma), g \neq 1\), has the length \(l(g) = 2l(t) + 1\) relative to the decomposition (1). Moreover, by virtue of the regularity of the operation of the group of operators \(\Gamma\)

\[ l(t(\alpha\gamma)^{-1}) = 2l(t) + 1, \]

if \(\alpha \neq e\). Hence, the decomposition (5) we need follows.