REGULAR APPROXIMATIONS OF RECURSIVE PREDICATES*

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Let there be established a fixed machine concept (computational process) equivalent to the concept of an arithmetic partial recursive function (PRF), as well as a measure of the complexity of computations on machines of this type, satisfying the Blum-Tseitin axioms (see, e.g., [1], Sec. 3). Let \( \Phi \) be a general recursive function (GRF), and \( S \) a recursive predicate whose computation complexity on machines of the indicated type grows more rapidly than \( \Phi \). We wish to investigate a sequence of machines \( M_1, M_2, \ldots, M_n, \ldots \) such that, given any \( n \), machine \( M_n \) computes \( S \) for natural numbers \( x \leq n \) with a complexity not greater than \( \Phi(x) \). We call this sequence a \( \Phi \)-bounded approximation of the recursive predicate \( S \). We define the complexity of this approximation as a function that sets in correspondence with every natural number \( n \) the length of the code of the \( n \)-th member of the approximation (the code length is defined axiomatically).

The complexity of bounded approximations of recursively enumerable predicates has been investigated in [2, 3]. In the present article we consider one of the possible refinements of the bounded approximation problem for recursive predicates and establish results pertaining to the complexity of these approximations. We analyze in detail the case in which restrictions on the complexity of computation cause the members of the approximating sequence to degenerate into finite automata.

1. Let \( \langle \xi \rangle \) be an admissible (in the sense of [4]) Gödel numbering of PRF's. Following [1], we set in correspondence with each PRF \( \xi \) a PRF \( \Phi_i \) (a function of computation complexity \( \xi_i \)) and a natural number \( i, i \) (length of the record of \( \xi_i \)), both satisfying the following axioms:

**Axiom 1.** Given any natural number \( i \), the graph of the function \( \Phi_i \) is a recursive set, and the condition \( \forall n (\xi_i(n) \equiv \xi_i(n)) \) is satisfied [an expression of the form \( \xi_i(n) \) stands as an abbreviation for the statement "The PRF \( \xi_i \) is applicable to the number \( n \)."

**Axiom 2.** Given any number \( n \), the set \( \{ i : i \leq n \} \) is finite (i.e., can be effectively specified in the form of a list).

We denote by \( h_i(n) \) the number of elements of the set \( \{ i : i \leq n \} \). We call a function \( \varphi \) a representing function of the recursive predicate \( S \) if \( \varphi \) is a GRF and \( \forall x (\varphi(x) = 0 \iff x \in S) \).

We consider the following axiom to be satisfied, in addition to Axiom 1, for the family of functions \( \langle \Phi_i \rangle \):

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Axiom 3. A GRF $\psi$ is realizable such that if $m$ is the index number of a representing function of a finite predicate, then

$$\exists j \forall x \left( (\psi_j(x) = 0 = \varphi_m(x) = 0) \& \Phi_j(x) \leq \psi(x) \right).$$

It is easily shown that Axiom 3 is independent of Axiom 1 and is satisfied, in particular, for the class of Turing machines if the number of steps in the operation of the machine or the number of tape cells used is adopted as the measure of computation complexity.

Let $\Phi$ be a GRF such that $\forall n \left( \Phi(n) > \psi(n) \right)$, and let $S$ be a recursive predicate whose computation complexity grows more rapidly than $\Phi$ [i.e., if $m$ is the index number of a representing function of $S$, then $\forall n \exists j \Phi_m(j) > \Phi(j)$]. Hereinafter we replace the phrase "is introduced as the notation for" by the symbol $\equiv$. Let

$$\alpha \equiv \forall i (i \in S \equiv \varphi_m(i) = 0) \& \Phi_m(i) \leq \psi(i).$$

We call a general recursive function $\varphi$ a $\Phi$-bounded approximation of a recursive predicate $S$ if $\forall n \alpha \varphi(n)$. We call a $\Phi$-bounded approximation $\varphi$ of a recursive predicate $S$ minimal if for any $n$

$$|\varphi(n)| = \min_{\alpha \varphi(n)} |i|. $$

Let $\varphi$ be a minimal $\Phi$-bounded approximation of the recursive predicate $S$, with $\mu_\varphi(n) = |\varphi(n)|$.

1.1. We show that there exists an algorithm that will construct for a given GRF $\Phi$ and recursive predicate $S$ a minimal $\Phi$-bounded approximation $\varphi$ of $S$. We fix the predicate $S$ and a natural number $m$. According to Axiom 3, it is possible to construct a natural number $j$ such that $\alpha \varphi(n, j)$. According to Axiom 2, the set $\{m: |m| < |j|\}$ is finite. Using Axiom 1, we can pick from this set the numbers $\ell$ for which the condition $\alpha \varphi(n, \ell)$ is fulfilled; from those we pick out the numbers $\ell', \ell''$ for which $|\ell'|$ is the smallest (of which there can be several). We set $\varphi(n)$ equal to the smallest of these numbers. It is evident from the foregoing considerations that $\varphi$ is a GRF and is a minimal $\Phi$-bounded approximation of $S$.

Using Axioms 1-3, we can show that there is a GRF $L$ such that, given any predicate $S$ and any $n, \mu_\varphi(n) \leq L(n)$.

On the other hand, given any unbounded nondecreasing GRF $\varphi$, it is possible to construct a recursive predicate $S$ with an unbounded truth domain such that

$$\forall m \exists n \\mu_\varphi(n) < \varphi(n).$$

1.2. Let $\varphi$ be a two-place GRF nondecreasing on the second argument. We say that a recursive predicate $S$ is $\varphi$-simple if

$$\exists n \forall i \exists m \mu_\varphi(m) < \varphi(n, m).$$