UNIQUENESS OF THE SOLUTION OF STATIONARY ACOUSTIC PROBLEMS FOR REINFORCED PLATES

B. P. Belinskii

A plate reinforced by ribs and placed in an absorbing acoustic medium is considered. The uniqueness of the solution of the stationary acoustic problem for this system is established. The connection of the uniqueness problem with the optical theorem is discussed.

The energy uniqueness principle for boundary-contact problems of acoustics is discussed in [1]. It is established that under rather general assumptions, the stationary acoustic problem in a plate-medium system with losses is identically zero in the absence of external sources. Following the ideology of this work, we prove that a plate in an acoustic medium reinforced by a rib or two perpendicular ribs belongs to the systems possessing these uniqueness properties. In the absence of absorption, it is not possible to prove that the field generated by zero sources is trivial. It is found, however, that in this case there are no waves radiating energy to infinity in the total field. This fact is directly related to the optical theorem.

We consider stationary processes in a system consisting of a plate \((-\infty < x, y < \infty, z = 0)\) performing bending motions and of the acoustic half space \((z > 0)\) next to it. The plate is reinforced by a rib \((x = 0, -\infty < y < \infty, z = 0)\). The dependence of processes on time \(\exp(-i\omega t)\) \((\omega\) is the frequency of oscillations) is omitted everywhere. The acoustic processes in the system we characterize by the pressure \(p(x, y, z)\), while the oscillations of the plate are characterized by the bending displacement \(\zeta(x, y)\). The pressure satisfies the Helmholtz equation

\[
(\Delta + \kappa^2)p = 0,
\]

where \(k\) is the wave number, while the displacement satisfies the Kirchhoff equation

\[
D (\Delta_0 \zeta - \rho \zeta) + \rho \zeta = 0 \quad (x \neq 0),
\]

where \(D\) is the cylindrical rigidity of the plate; \(\kappa\), wave number of its bending oscillations; and \(\Delta_0\), Laplace operator in the coordinates \(x, y\). Here the connection of the pressure at the surface of the plate with its displacement is given by the adherence condition \(\zeta = -\frac{1}{Z_0 \rho} \frac{\partial p}{\partial z} |_{z=0}\), where \(\rho_0\) is the density of the medium. We model the reinforcing rib by a rod of rectangular cross section which is capable of bending and torsion oscillations. Conditions that it be fastened to the plate are given on the line \((x = 0, -\infty < y < \infty)\) and have the form [2]

\[
-D \left[ \zeta_{xx} + (2-\sigma) \zeta_{yx} \right] = -i \omega Z_F \zeta,
\]

\[
D \left[ \zeta_{xx} + \sigma \zeta_{yy} \right] = -i \omega Z_M \zeta_x,
\]

where \(\sigma\) is the Poisson coefficient of the material of the plate, the symbol \([\psi]_x\) denotes the jump of the function \(\phi(x, y)\) on passing across the rib \(x = 0\), and \(Z_F\) and \(Z_M\) are, respectively, the force and momentum impedances of the rib.

Here $E$ is Young's modulus of the rib; $I$, moment of inertia of the cross section of the rib relative to an axis passing through the middle of its lower edge; $K$, torsional rigidity of the cross section; $\rho$, density of the rib; and $b$ and $H$, its height and thickness.

As is evident, in correspondence with the usual scheme of proving uniqueness theorems for linear problems, we assume that external volume forces in the fluid, surface forces acting on the plate, and forces and moments applied directly to the rib are absent.

A solution of this problem is sought according to the principle of limiting absorption. Namely, the wave number is first assumed complex ($0 < \arg k < \pi/2$), and the corresponding field $p(x, y, z, k)$ is constructed; in the absence of absorption the field is taken equal to the limit

$$
\lim_{\Im k \to 0} p(x, y, z, \Re k + i \Im k).
$$

Our objective is to prove that this problem for complex $k$ has only the trivial solution. We distinguish some volume of the medium $\Omega$ adjacent to the plate along an area $S$. The remainder of its surface we denote by $\Sigma$. Suppose that there is a segment of the reinforcing rib ($x = 0, y_1 < y < y_2$) in the area $S$. Together with the state of the system characterized by the pressure $p$ and the displacement $\zeta$, we consider the state characterized by the pressure $p^*$ and the displacement $\zeta^*$ (the asterisk denotes the complex conjugate). Since we assume that damping occurs only in the medium, relations (2) and (3) also hold for the state $p^*, \zeta^*$. The Helmholtz equation (1) is an exception; here it is necessary to replace $k$ by $k^*$:

$$
(\Delta + k^*\kappa^2) p^* = 0.
$$

We invoke Green's formula for the Laplace operator

$$
\frac{1}{\varepsilon \rho^* \omega} \int_{\Omega} \left( p^* \frac{\partial \rho^*}{\partial n} - \rho \frac{\partial p^*}{\partial n} \right) d\Omega = \frac{1}{\varepsilon \rho^* \omega} \int_{\Sigma} \left( p^* \Delta \rho - \rho \Delta p^* \right) d\Sigma,
$$

where $n$ is the unit outer normal. Considering the Helmholtz equations (1) and (4), we find

$$
\frac{1}{\varepsilon \rho^* \omega} \int_{\Omega} \left( p^* \frac{\partial \rho^*}{\partial n} - \rho \frac{\partial p^*}{\partial n} \right) d\Omega = \frac{1}{\varepsilon \rho^* \omega} \int_{\Sigma} \kappa^2 |p|^2 d\Sigma.
$$

The left side of the last equality gives the flux of acoustic energy through the surface of the volume $\Omega$, while the right side is equal to the energy absorbed in the region taken with opposite sign. The integral over the area $S$ can be transformed using the adherence condition to the form

$$
- \frac{\omega}{2} \operatorname{Im} \left. \int_{S} p^* \right|_{Z=0} ^{Z=Z_0} dS.
$$

We further invoke Green's formula for the biharmonic operator [3]

$$
\operatorname{Im} \frac{\omega}{2} \int_{L \cup S \cup L} \left[ \hat{f}(\xi) \zeta + M(\xi) \frac{\partial \xi}{\partial n} \right] dL = \operatorname{Im} \frac{\omega^2}{2} \int_{S} \Delta^2 \zeta dS.
$$

Here $L$ is the smooth boundary of $S$, $L_z = (x = \pm 0, y_1 < y < y_2)$,

$$
\hat{f}(\xi) = \frac{\partial}{\partial n} \left( \frac{\partial}{\partial n} \Delta \zeta + (1-\xi) \left( \frac{\partial^2 \zeta}{\partial S^2} - \frac{\partial}{\partial n} \frac{\partial \zeta}{\partial n} \right) \right),
$$

$$
M(\xi) = \frac{\partial}{\partial n} \left( \Delta \zeta - (1-\xi) \left( \frac{\partial^2 \zeta}{\partial S^2} + \frac{1}{R} \frac{\partial \zeta}{\partial n} \right) \right).
$$