OPTICAL THEOREM FOR THE SCATTERING OF WAVES
IN AN ELASTIC PLATE

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The scattering of waves in an elastic plate is investigated. A generalization of the optical theorem is found for the case of a noncompact scatterer. The formulas obtained can be used for indirect control of the correctness of computing diffraction fields.

An identity is proved for the direction diagram of the field scattered during the diffraction of waves by a noncompact inhomogeneity in an elastic plate. It includes the interpretation of the law of conservation of energy and may be useful in indirect control of the results of computing diffraction fields.

In scattering theory it is known that the effective cross section of scattering by a compact obstacle is proportional to the real part of the direction diagram at zero angle. This assertion is a consequence of the unitarity of the scattering operator and is known as the optical theorem [1, 2].

It is shown in [3] how the optical theorem changes in the case where in an acoustic medium there is a thin infinite plate together with the obstacle causing the scattering. Namely, it is necessary to consider a new scattering channel - the propagation of energy along the plate and the liquid layer next to it. Similar considerations for waves in a semi-infinite elastic medium are contained in [4], where there is a bibliography on this problem for an infinite, elastic medium. In particular, it is found that the scattering cross section is formed from the sum of the sections for longitudinal and transverse waves.

In the present work analogous formulas are discussed for the scattering of waves in a thin vibrating plate. In a technical sense, the work is related to [3], and we shall therefore omit involved computations which are analogous to those carried out there.

1. We consider the bending vibrations of an infinite, elastic plate. We first suppose that its homogeneity is interrupted only in some finite region (an inclusion of homogeneous material, a cut, etc.). On this region there is incident a plane wave \( \xi_0 = \exp(\imath \kappa \cos(\varphi - \varphi_0)) \) (\( \kappa \), wave number of flexure vibrations; \( (r, \varphi) \), polar coordinates; and \( \varphi_0 \), angle of incidence). In the far-field zone (\( \kappa r \gg 1 \)) diverging cylindrical waves are formed, and the total field has the form

\[
\zeta = \zeta_0 + \sqrt{\frac{2\pi}{x_0}} \exp(\imath \kappa r - \imath \varphi_0/4) \psi(\varphi, \gamma_\circ),
\]

where \( \psi(\varphi, \gamma_\circ) \) is the scattering diagram.

We introduce the integral of energy flux \( \Pi \) through a circle of large wave radius. Because of the absence of sources inside the circle

\[
\lim_{r \to \infty} \oint_{\partial B} \Pi d\ell (\Pi, \vec{e}_r) = 0.
\]

Here \( \vec{e}_r \) is the position vector, and the expression for the flux of energy for the displacement \( \zeta \) can be borrowed from [5]. Applying the method of descent to find the limit (1.2), we obtain

where we have introduced the notation

$$ W + W_0 \frac{4\pi}{\lambda} \text{Re} \psi(q_0, q_0) = 0, \quad (1.3) $$

Thus, for the effective scattering cross section $\sigma = W/W_0$ we find

$$ \sigma = \frac{2\pi}{\lambda} \int_{-\infty}^{\infty} |\psi(q, q_0)|^2 \, dq, \quad (1.6) $$

and the optical theorem assumes the form

$$ \sigma = -\frac{4\pi}{\lambda} \text{Re} \psi(q_0, q_0). \quad (1.7) $$

The formulas found have a clear physical interpretation; the energy carried away by the cylindrical wave is taken from the plane incident wave during the process of their interaction.

2. We now consider a semi-infinite plate (-\infty < x < \infty, y < 0) with a finite inhomogeneity. We assume that the boundary (y = 0) is fixed. The oscillations of the plate are excited by the incidence of the plane wave mentioned above. Here a plane wave and a certain wave of inhomogeneous character are reflected from the boundary, and in the far-field zone a cylindrical diverging wave is formed:

$$ z = \exp(i \alpha x \cos \phi_0 + y \sin \phi_0) + R \exp(i \alpha x \cos \phi_0 - y \sin \phi_0), \quad (2.1) $$

$$ R = \frac{i \sin \phi_0 - \sqrt{1 + \cos^2 \phi_0}}{i \sin \phi_0 + \sqrt{1 + \cos^2 \phi_0}}, \quad T = \frac{-2i \sin \phi_0}{i \sin \phi_0 + \sqrt{1 + \cos^2 \phi_0}}. $$

In this case the optical theorem has the form

$$ \sigma = -\frac{4\pi}{\lambda} \text{Re} \left( R^* \psi(-q_0, q_0) \right), \quad (2.2) $$

and the total scattered power is computed by formula (1.6) as before; here the asterisk denotes the complex conjugate.

Suppose now that the edge of the plate is free. A surface wave (a proper process) can then propagate along it:

$$ \xi_c = d_\pm(i \alpha_c x + \sqrt{\alpha_c^2 + \omega^2} y) + \xi_c \exp \left( \pm i \alpha_c x + \sqrt{\alpha_c^2 - \omega^2} y \right) \quad (x \geq 0). $$

Here $\alpha_\pm$ are the amplitudes of surface waves propagating in the directions $x \equiv 0$, respectively; $\alpha_c$, their wave number; $\alpha_c = \kappa \sqrt{\gamma}$; and $t$, root of the dispersion equation

$$ \sqrt{t^2 + 1} \left( \frac{t + \frac{1}{1 - \delta}}{t - \frac{1}{1 - \delta}} \right) = \sqrt{t^2 - 1} \left( \frac{t + \frac{1}{1 - \delta}}{t - \frac{1}{1 - \delta}} \right)^2, \quad \text{if} \quad t \in (1, \frac{1}{1 - \delta}), \quad (1.4) $$

$$ \delta = \frac{t + \frac{1}{1 - \delta}}{t - \frac{1}{1 - \delta}}. $$

In the present case a term is added to the effective cross section (1.6) connected with the scattering channel described.