JUSTIFICATION OF AN ASYMPTOTIC FORMULA
FOR SOLUTIONS OF THE PERTURBED KLEIN – FOCK – GORDON EQUATION

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A problem with periodic boundary conditions is considered for the Fock–Klein–Gordon equation perturbed by a small nonlinear operator $\varepsilon R[\varepsilon t, u, u_x, u_{xx}]$:

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u &= \varepsilon R[\varepsilon t, u, u_x, u_{xx}] ; \\
\left. u \right|_{t=0} &= a \cos x ; \quad \left. \frac{\partial u}{\partial x} \right|_{t=0} = a \sin x ; \\
\omega^2 &= c^2 + m^2 ; \quad u(x+2\pi) = u(x).
\end{align*}
$$

It is shown that under certain conditions the solution exists and is close to the known asymptotic solution on the interval $0 \leq t \leq l/\varepsilon$.

Various asymptotic methods for nonlinear partial differential equations have recently become popular. Both a variational formulation of these methods [1] and an approach based on series expansions are possible [2–4]. The problem of justifying the asymptotic formulas in the general case is very complicated. In the present work a means of such justification is proposed for the Klein–Fock–Gordon (KFG) equation

$$
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u = \varepsilon R[u], \quad \varepsilon << 1,
$$

perturbed by some small nonlinear operator $\varepsilon R[u]$, where $m$ and $c$ are constants.

1. Formulation of the Problem and Derivation of a System of Integral Equations

We consider the mixed problem

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u &= \varepsilon R(u_x, u_{xx}, \varepsilon t), \\
\left. u \right|_{t=0} &= a \cos x ; \quad \left. \frac{\partial u}{\partial x} \right|_{t=0} = a \sin x,
\end{align*}
$$

where $\omega^2 = c^2 + m^2$, $m \neq 0$, with boundary conditions that are $2\pi$–periodic in $x$,

$$
u(x+2\pi) - u(x)
\]

We suppose that the function $R$ is an analytic function of the variables $u, u_x, u_{xx}$ and is a twice continuously differentiable function of $\varepsilon t = \tau$. The present problem represents the problem of the deformation of a sinusoidal wave under the action of the perturbation $\varepsilon R$. A formal asymptotic expansion of solutions of (1.1) (as $\varepsilon \to 0$) was found in [5]. The leading term of this expansion has the form

$$
\begin{align*}
\left. u \right|_{t=0} &= a \cos x.
\end{align*}
$$

Here the amplitude \( a \) and the phase \( z = x - \omega t + b \) satisfy the equations

\[
\frac{da}{d\varepsilon} = (2\omega)^{-1} f(a), \quad \frac{dz}{d\varepsilon} = (2\omega)^{-1} g_t(a),
\]

where

\[
\begin{align*}
\tilde{f}(a) &= \frac{\Gamma}{a} \int R(u_d, u_{ox}, u_{oxx}, \tau) \sin z \, dz, \\
\tilde{g}_t(a) &= \frac{\Gamma}{a} \int R(u_d, u_{ox}, u_{oxx}, \tau) \cos z \, dz.
\end{align*}
\]

The question arises, however, of whether the first approximation is close to the exact solution over times of order \( \varepsilon^{-1} \) as \( \varepsilon \to 0 \). This question is the more natural, since the very existence of a solution \( u \) of problem (1.1), (1.1') is far from obvious.

In the present work it is shown that a solution of the problem exists on a time interval \( [\varepsilon] - [0, \varepsilon^{-1}] \) for sufficiently small \( \varepsilon \). Moreover, it is an analytic function of \( x \) in some strip \( \Gamma_k(\varepsilon) = \{ x : |x| < \varepsilon \} \), and inside this strip it satisfies the estimate

\[
u = \frac{a \cos z + \varepsilon(\psi(x,t), \max_{\sigma} |\psi(x,t)| < K.\]

The positive constants \( l \) and \( K \) depend only on the maximum modulus of the function \( R(u, u_x, u_{xx}, \tau) \) in the region \( |u| < \alpha \varepsilon^\sigma, |u_x| < \alpha \varepsilon^\sigma, |u_{xx} < \alpha \varepsilon^\sigma, \tau \leq 1 \).

We proceed to the solution of the problem. We represent the function \( u(x, t) \) in the form

\[
u = u_t + \varepsilon u = a \cos z + \varepsilon(\psi(x,t),
\]

and in Eq. (1.1) we go over to the variables \( (z, t) \), \( z = x - \omega t + b(\tau) ; b(\tau) \) satisfies (1.3). For the function \( v(z, t) \) we obtain the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
\mu_t - 2\omega \mu_x + m \mu + m^2 \nu_{tx} + m^3 \nu_{xx} + 2 m^5 \nu_{xxx} - 2 e \delta \nu \nu_{tx} \\
\nu_t + 2 e \delta \nu = 0, \quad \nu_{t} = 0, \quad \nu(z + 2\delta) = \nu(z)
\end{array} \right.
\end{align*}
\]

where

\[
\Phi_k = R(u_0, u_{ox}, u_{oxx}, \tau) - 2 \omega u_{ox}, \Phi_k = \Phi_k(\omega - 2 \delta) e^{\delta \mu} - e^{\delta \mu} \mu - 2 \omega u_{ox} \delta - u_{ox} + \varepsilon^{-1} (R(u_0, u_{ox}, u_{oxx}, \tau) - R(u_0, u_{ox}, u_{oxx}, \tau) - e^{\delta \mu} u_{ox} u_{oxx} + e^{\delta \mu} u_{xx})
\]

To solve problem (1.7), we use the Galerkin method [6]. We choose a segment of the Fourier series

\[
\psi(z, t) = \sum_{k=-N}^{N} \psi_k(t) e^{ikz}, \quad \psi_k = \psi_k
\]

and we substitute it into (1.7), imposing the standard conditions of orthogonality with respect to the system of functions \( \{ e^{ikz} \} \), \( k = 0 \neq 1, \pm 2, \ldots, \pm N \) [6]. For the Fourier coefficients \( v_k(t) \) we obtain the system of ordinary differential equations

\[
\begin{align*}
\frac{dv_k}{dt} + 2 \omega k e^{ikz} v_k + m^4 (1 - k^2) v_k - \Phi_k e^{ikz} v_k - 2 e \delta \nu \nu_{tx} \\
\frac{dv_k}{dt} = 0.
\end{align*}
\]

For brevity we shall sometimes omit the index \( N \) below. System (1.9) can easily be reduced to a system of integral equations. We have

\[
\begin{align*}
v_k(t) = \frac{1}{\delta} \int_0^t \int G_k(t, s) (\Phi_k e^{ikz} v_k(s, \varepsilon) - 2 e \delta \nu(s) v_k) d \varepsilon dz,
\end{align*}
\]

where \( G_k(t, s) = (\exp \lambda_a(t-s) - \exp \lambda_a(t-s)) (2 \mu + m^2 - 2 e \delta \nu) e^{ikz} \). Integrating the last term in (1.11) by parts, we finally obtain

\[
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\]