All the subgroups between the special linear groups \( SL(n, o) \) and \( SL(n, o') \) are described in the following two cases: 1) \( o \) is a real-closed field, and \( o' \) is its algebraic closure; 2) \( o \) is a Euclidean ring, and \( o' \) is its quotient field.

Let \( o \) be a subring with identity element of the ring \( o' \), and let \( n \) be a natural number. In the present article we describe the subgroups contained between \( SL(n, o) \) and \( SL(n, o') \) for certain \( o \) and \( o' \).

Let \( k \) be a real-closed field, and let \( K \) be its algebraic closure. It turns out that the subgroups between \( SL(n, k) \) and \( SL(n, K) \) are exhausted by the subgroups

\[
H_{nm} = SL(n, K) \cap \mathfrak{G}(e^n) \cdot GL(n, k),
\]

where \( e \) is the primitive root of degree \( 2n \) of unity and \( m \) spans the set of all divisors of \( n \). In particular, the lattice of these subgroups is isomorphic to the lattice of divisors of \( n \), and subgroup \( H_{n1} \) is maximal in \( SL(n, K) \).

The second case in which it is possible to solve the problem completely occurs when \( o \) is a Euclidean ring and \( o' \) is its quotient field. In this case the intermediate subgroups have a one-to-one correspondence with the intermediate subrings, and as the latter have an explicit description, the following result is obtained: The intermediate subgroups between \( SL(n, o) \) and \( SL(n, o') \) are exhausted by the subgroups \( SL(n, o_T) \), where \( o_T \) is the ring of quotients of ring \( o \) whose denominators are divisible by prime factors of \( n \) and \( m \) runs through all possible sets of prime elements of ring \( o \). This formally includes the description of the maximal subgroups between \( SL(n, Z) \) and \( SL(n, Q) \) in [1], but the procedure used in [1] gives more than is actually stated in that paper, namely if \( o \) is the maximal norming subring of field \( k \), then subgroup \( SL(n, o) \) is maximal in \( SL(n, k) \).

1. Let \( k \) be a real-closed field, and let \( K \) be its algebraic closure. We invoke without discussion the results of the classical theory of formally real fields of Artin and Schreier and the facts concerning elementary transformations of the type of Lemma 3 in [1]. Moreover, we require the following definition from [1]: Two elements of group \( G \) are said to be equivalent on the module of subgroup \( H \) if each one generates together with \( H \) the same subgroup.

**Lemma 1.** Let

\[
z = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad a \in K \setminus k.
\]

Then

\[
\mathfrak{G}(z, SL(2, k)) = SL(2, K).
\]
Proof. Let \( c_0 = a_0 + b_0i \); \( a_0, b_0 \in k \). Here and elsewhere \( i \) denotes the root of the polynomial \( x^2 + 1 \). We fix an arbitrary element \( c = a + bi \), \( a, b \in k \), \( b \neq 0 \). The signs of \( b \) and \( b_0 \) are assumed to be the same (otherwise we would consider \( z^{-1} \) instead of \( z \)). We have

\[
z \sim \begin{pmatrix} \sqrt{b_0} & 0 \\ 0 & \sqrt{b_0} \end{pmatrix} z \begin{pmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} 1 & -a_0/b_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod SL(2, k).
\]

(1)

If \( a \neq 0 \) and \( a^2 + b^2 = 1 \), then

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \mod SL(2, k).
\]

(2)

Here and elsewhere \( \overline{c} \) denotes \( a - bi \) if \( c = a + bi \), \( a, b \in k \). The latter equivalence is based on the fact that \( \alpha c - \beta \overline{c} = -1 \) for certain \( \alpha, \beta \in k \); hence the matrix \( \begin{pmatrix} 0 & c \\ -\overline{c} & 1 \end{pmatrix} \) can be translated by elementary transformations over \( k \) into \( \begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix} \). Inasmuch as the cofactors in (2) belong to \( \Gr \{ z, SL(2, k) \} \) (we use arbitrary \( c \)), \( \begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix} \) belongs to the same group. Moreover,

\[
\begin{pmatrix} 1 + i & 0 \\ 0 & 1 - i \end{pmatrix}^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

(3)

The matrices derived in (1), (2), and (3), given different \( c \), generate together with \( SL(2, k) \) the group \( SL(2, K) \), and this completes the proof of the lemma.

**Lemma 2.** If

\[
y_{sm}^s = c + c_0c_{sm} \subseteq SL(n, K), \ s \neq m, c_0 \subseteq K \setminus k,
\]

then

\[
\Gr \{ y_{sm}^s, SL(n, k) \} = SL(n, K).
\]

The proof of the lemma is analogous to the preceding proof and rests on the fact that \( SL(n, K) \) is generated by all its diagonal matrices and matrices of the form \( e + ce_i, l \neq j, c \in K \). It should be pointed out that an arbitrary diagonal matrix from \( SL(n, K) \) can be decomposed into a product of diagonals for which only two contiguous elements along the diagonal can differ from the corresponding elements of the unit matrix.

**Lemma 3.** If two diagonal elements of the diagonal matrix \( z \in SL(n, K) \) are not proportional over \( k \), then

\[
\Gr \{ z, SL(n, k) \} = SL(n, K).
\]

Proof. Indeed, conjugating by means of \( z \) the proper matrix \( e + e_{sm}, s \neq m, \) we arrive at the premise of Lemma 2.

**Lemma 4.** Let \( T(n, K) \) denote the group of nondegenerate triangular matrices with zero in the upper apex and coefficients of \( K \), and let \( z \in T(n, K) \). Let us assume that not all the coefficients \( z \) are pairwise proportional over \( k \). Then there exist unitriangular matrices \( t_1, t_2, t_3, t_4 \in T(n, K) \cap SL(n, k) \) and a diagonal matrix \( \tau \in SL(n, k) \) such that

\[
(t_1zt_4)^{-1} (t_3zt_2)^t \in H_n = H_{n1} \quad \text{(see Introduction)}.
\]

Proof. Clearly, the matrix \( (t_1zt_4)^{-1} (t_3zt_2)^t \) is unitriangular and belongs to \( H_n \) only if all its coefficients are in \( k \). If one of the two cells of dimension \( n - 1 \) in the upper left corner of matrix \( z \) or in the lower right corner satisfies the conditions of the lemma for dimension \( n - 1 \), all that is required is to apply induction (the transformations of the cells can be logically expanded to transformations of the matrices) and to examine the case \( n = 2 \). But if such is not the case, i.e., if the indicated cells all the elements are pairwise