ON A CHARACTERISTIC PROPERTY
OF THE LORENTZ SPACE IN THE CLASS
OF SYMMETRIC SPACES

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We establish a criterion for the equivalence of norms in a symmetric space and in the Lorentz space, in particular, a criterion for the equivalence of norms in an Orlicz space and in the Lorentz space.

By a symmetric space we mean [1] the space E of functions, measurable on [0, 1], satisfying the conditions: 1) if $x(t) \in E$ and if the functions $|x(t)|$ and $|y(t)|$ are equimeasurable, then $y(t) \in E$ and $\|y(t)\|_E = \|x(t)\|_E$; 2) if $|y(t)| \leq |x(t)|$ and $x(t) \in E$, then $y(t) \in E$ and $\|y(t)\|_E \leq \|x(t)\|_E$.

The function $\varphi(t) = \|x_0(s)\|_E$ (me = t), where $x_0(t)$ is the characteristic function of the set $e \subseteq [0, 1]$, is called the fundamental function of the space E. The function $\varphi(t)$ is nondecreasing [1] and, with accuracy up to an equivalent norm, $\varphi(t)$ is concave [9]. Hence, in what follows we shall always assume that $\varphi(t)$ satisfies these conditions.

Consequently, $\varphi'(t)$ exists almost everywhere. In addition we shall assume that $\lim_{t \to 0} \varphi(t) = 0$. This is equivalent to the condition that $E \subseteq L_\infty$ (see [1]).

Let $\Lambda_\varphi$ denote the Lorentz space and let $M_\varphi$ denote the Marcinkiewicz space. The norms in $\Lambda_\varphi$ and $M_\varphi$ are defined by:

$$|x(t)|_{\Lambda_\varphi} = \int_0^1 x^*(t) d\varphi(t), \quad \|x(t)\|_{M_\varphi} = \sup_{0 \leq t \leq 1} \frac{\varphi(t)}{t} \int_0^t x^*(t) dt,$$

where $x^*(t)$ is a nonincreasing rearrangement of $|x(t)|$.

The spaces $\Lambda_\varphi$ and $M_\varphi$ have the same fundamental function $\varphi(t)$.

If E is a symmetric space and $\varphi(t)$ is the fundamental function of E, then $\Lambda_\varphi \subseteq E \subseteq M_\varphi$ and this embedding is continuous [1].

Let $E'$ denote the space associated to E, that is, the space of functions $y(t)$ such that

$$\|y(t)\|_{E'} = \sup_{0 \leq t \leq 1} \left| \int_0^t y(t) x(t) dt \right| < \infty.$$

The space $E'$ is symmetric. The fundamental function of $E'$ is the function $\varphi^*(t) = t/\varphi(t)$, where $\varphi(t)$ is the fundamental function of E.

We denote by $E''$ the space associated to $E'$.

1. Our main result is Theorem 1 which gives a characteristic property of the Lorentz space in the class of symmetric spaces.

THEOREM 1. Let E be a symmetric space and let $\varphi(t)$ be its fundamental function.
1) If \( \varphi'(t) \) belongs to \( E' \) then \( E \) is separable and the norms of the spaces \( E, E'' \) and of the space \( \Lambda_{\varphi} \) are equivalent. The bound for the norms is as follows:

\[
x(t)_{E} \leq \|x(t)\|_{\Lambda_{\varphi}} \leq \|\varphi'(t)\|_{E'} \|x(t)\|_{E''} \leq \|\varphi'(t)\|_{E'} \|x(t)\|_{E}.
\]

(1)

2) If the spaces \( E \) and \( \Lambda_{\varphi} \) have equivalent norms, then \( \varphi'(t) \in E' \).

**Proof.** Let us prove the first part of the Theorem. The space \( E \) is continuously embedded in \( E'' \) and

\[
\|x(t)\|_{E''} \leq \|x(t)\|_{E}.
\]

(2)

The space \( E'' \) is continuously embedded in the Lorentz space \( \Lambda_{\varphi} \). In fact, since the space associated to a symmetric space is a symmetric space and, by hypothesis, \( \varphi'(t) \in E' \), then for any function \( x(t) \in E \) we have

\[
\|x(t)\|_{\Lambda_{\varphi}} = \int_{0}^{1} x'(t) \varphi'(t) dt \leq \|\varphi'(t)\|_{E'} \|x(t)\|_{E''}.
\]

(3)

This inequality means that the space \( E'' \) is continuously embedded in \( \Lambda_{\varphi} \). It was shown in [1] that \( \Lambda_{\varphi} \) is continuously embedded in \( E \) and that

\[
\|x(t)\|_{E} \leq \|x(t)\|_{\Lambda_{\varphi}}.
\]

(4)

The inequality (1) follows from (2), (3), and from (4) and, hence, the norms in the spaces \( E, E'' \) and \( \Lambda_{\varphi} \) are equivalent. The separability of \( E \) follows from the equivalence of the norms in \( E \) and \( \Lambda_{\varphi} \) and from the separability of \( \Lambda_{\varphi} \).

To prove the second part of the theorem it is sufficient to show that \( \varphi'(t) \in M_{\Phi}^* \) because the Marcinkiewicz space \( M_{\Phi}^* \) (see [3]) is associated to \( \Lambda_{\varphi} \). Let us show that \( \varphi'(t) \in M_{\Phi}^* \). Since \( \Phi^*(t) = t/\varphi(t) \), we have that

\[
\|\varphi'(t)\|_{M_{\Phi}^*} = \sup_{0 < t < 1} \left( \int_{0}^{1} \frac{\varphi^*(t)}{t} \varphi'(t) dt \right) = \sup_{0 < t < 1} \left( \frac{1}{\varphi(t)} \int_{0}^{1} \varphi'(t) dt \right) = 1,
\]

that is, \( \varphi'(t) \in M_{\Phi}^* \), and the theorem is proved.

Suppose that \( M(u) \) is an \( N \)-function and that \( N(v) \) is its complementary function (see [4], p. 22). Let \( L_{M}^* \) denote the Orlicz space generated by the \( N \)-function \( M(u) \). A norm in \( L_{M}^* \) is given by

\[
\|x(t)\|_{M} = \sup \left\{ |x(t) y(t)| \right\},
\]

where the supremum is taken over all functions \( y(t) \) satisfying the condition

\[
\int_{0}^{1} N[y(t)] dt \leq 1.
\]

The fundamental function of \( L_{M}^* \) is \( \varphi_{M}(t) = tN^{-1}(1/t) \) (see [4], p. 88).

Let \( q(v) \) denote the right derivative of the \( N \)-function \( N(v) \).

**COROLLARY 1.** If the function

\[
\varphi_{M}(t) = N^{-1}(1/t) - \frac{1}{t} N^{-1}(1/t)
\]

belongs to the Orlicz space \( L_{N}^* \), then the space \( L_{M}^* \) is separable and the norms in \( L_{M} \) and \( \Lambda_{\varphi_{M}} \) are equivalent.

**Proof.** We denote the space \( L_{M}^* \) by \( E \). The pairs of spaces \( E' \) and \( L_{N}^* \) and \( E'' \) and \( L_{M}^* \) have equivalent norms and the following holds:

\[
\frac{1}{\rho} \|y(t)\|_{N} \leq \|y(t)\|_{E'} \leq \|y(t)\|_{N} \quad (y(t) \in L_{N}^*);
\]

\[
\frac{1}{\rho} \|z(t)\|_{M} \leq \|z(t)\|_{E'} \leq 2 \|z(t)\|_{M} \quad (z(t) \in L_{M}^*).
\]

By hypothesis \( \varphi_{M}(t) \in L_{N}^* \). Hence \( \varphi_{M}(t) \in E' \). When we apply Theorem 1 we obtain the assertion in the corollary.

2. The authors of [5] considered the question of the convergence to zero of the family of linear functionals.