STRUCTURE OF HOMOTHETIC LINEARLY SEPARABLE SETS
IN $n$-DIMENSIONAL EUCLIDEAN SPACE. III*

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UDC 514.17 + 517.11

We consider the structure of a finite collection of pairwise nonintersecting homothetic centrally symmetric convex sets in $\mathbb{R}^n$ that form a system of linearly separable sets.

7. STRUCTURE OF ARBITRARY CENTRALLY SYMMETRIC CONVEX FIGURES THAT GENERATE LINEARLY SEPARABLE SETS

In this section, we explore the structure of arbitrary plane (not necessarily smooth) centrally symmetric convex figures that generate linearly separable sets (LSS). We somewhat modify the definitions of Secs. 3-5, omitting the smoothness requirement for the convex figures. The notation remains as before. It is easy to see that all the results of Secs. 3, 4 remain valid without the assumption of smoothness of the convex figures. The results are proved similarly, replacing the tangent with the supporting line (in all propositions, except Corollary 3.3 and Lemma 4.1).

The class of figures $S$ is defined as in Sec. 3, without assuming smoothness of the figures $f$ from this class. We denote by $S \subset S$ the class of all nonsmooth figures from $S$ (i.e., figures from $S$ whose boundaries have corner points). If the boundary is nonsmooth, the modified corollary is stated in the following form.

COROLLARY 3.3. Let $f_1$ be an arbitrary figure from the class $S$, $A$ an arbitrary point of the boundary $\Gamma(f_1)$, and $\pi(A)$ the vertical angle to the angle formed by the extreme supporting lines to $f$ at the point $A$ that contains the figure $f_1$. Then for any point $z \in \pi(A)$ there exists a figure $f_2 \subset \pi(A)$ tangent to $f_1$ at the point $A$ such that $Z \in \Gamma(f_2)$ and $f_2 \in \{f_1, f_2\}_S$.

Remark. If $A$ is not a corner point, then the angle $\pi(A)$ becomes the half-plane $\pi$ defined by the tangent to $f_1$ at the point $A$ which does not contain the figure $f_1$.

We will now show how to restate the existence problem of tangent figures for the new class: let $f_1 \in S$ and the points $A, B \in \Gamma(f_1)$; do there always exist $f_2, f_3 \in \{f_1, f_2, f_3\}_S$ that are tangent to one another and tangent to $f_1$ at the points $A$ and $B$ respectively? For what $k$ can these figures be constructed? Denote by $\pi(A)$ and $\pi(B)$ the vertical angles introduced in the modified version of Corollary 3.3. It is easy to note that such figures $f_2$ and $f_3$ do not exist for any $k$ if $\pi(A) \cap \pi(B) = \emptyset$.

Now assume that the angles $\pi(A)$ and $\pi(B)$ intersect: $\pi(A) \cap \pi(B) = M$. Consider the family of figures $f_2(k) = \{H_A^{-k}(f_1)\}$ with $k$ varying from 0 to $+\infty$. Clearly, $f_2(k) \cap M = \emptyset$ for small $k$ and $f_2(k) \cap M \neq \emptyset$ when $k$ is sufficiently large. Thus there exists $k_0$ such that $f_2(k_0)$ is tangent to the set $M$, i.e., the figures $f_2(k_0)$ and $M$ intersect only at boundary points.

LEMMA 4.1. Let $f_1 \in S$ and let $A, B$ be two points on the boundary of the figure $f_1$, such that the angles $\pi(A)$ and $\pi(B)$ defined in the modified version of Corollary 3.3 intersect, $k$ an arbitrary value from the interval $(k_0, +\infty)$. Then there exists a minimal value of the similitude coefficient $n = n(k)$ of the figures $f_3$ and $f_1$ for which the figure $f_3(n) = H_B^{-n}(f_1)$ is tangent to the figure $f_2(k) = H_A^{-n}(f_1)$.

The proof of the modified propositions is similar to the proof of the original propositions.

*For Parts I and II, see Nos. 1 and 2 of this journal (1992).

Consider an arbitrary figure \( \Phi_1 \in \mathcal{S} \), which is not a polygon (a polygon is a closed polygonal line with any number of vertices). This means that there exists an arc \( \sim FB \subseteq \Gamma(\Phi_1) \) that does not coincide with the segment \([FB]\), where \( B \) is a corner point of \( \Gamma(\Phi_1) \). Denote the extreme supporting lines at the point \( B \) by \( l_2 \) and \( l_2' \), where \( l_2' \) is obtained as the limit position of the supporting lines at the point \( M \in \sim FB \) as \( M \to B \) and \( l_2 \) is obtained as the limit position of the supporting lines at the point \( N \) as \( N \to B \) along the curve \( \Gamma(\Phi_1) \) on the second arc of this curve that does not coincide with the arc \( \sim FB \). Choose a point \( A \) on the arc \( \sim FB \) so that the arc \( \sim AB \) is not the segment \([AB]\). Denote by \( l_1 \) the supporting line to the figure \( \Phi_1 \) at the point \( A \). This supporting line is unique, because the point \( A \) may be chosen as a noncorner point of the curve \( \Gamma(\Phi_1) \). Let \( x = l_1 \cap l_2 \) and \( \hat{x} = l_1 \cap \hat{l}_2 \) (Fig. 21). Construct the figure \( \Phi_3 = H_B^{-n_1}(\Phi_1) \) so that \( \text{int}(\Phi_3) \cap l_1 \neq \emptyset \). It is easy to see that for \( n > n_1 \),

\[
\text{int}(H_B^{-n}(\Phi_1)) \cap l_1 \neq \emptyset.
\]

By Sec. 4 and the remarks at the beginning of this section, for any \( n > n_0 \) there exists \( k(n) > k_0 \) such that the figures \( \Phi_3(n) \) and \( H_A^{-k(n)}(\Phi_1) \) are externally tangent. Varying the value of \( n \), we can ensure that the tangency point of the figures \( \Phi_3(n) \) and \( \Phi_2(n) \) (that lies on the segment \([O_2O_3]\) joining the centers of symmetry of \( \Phi_2(n) \) and \( \Phi_3(n) \)) is not a corner point of the curves \( \Gamma(\Phi_2(n)) \) and \( \Gamma(\Phi_3(n)) \). Denote this tangency point by \( C \). Let \( \Phi_2 = \Phi_2(n) \), \( \Phi_3 = \Phi_3(n) \), \( l_3 \) is the common internal tangent to the figures \( \Phi_2 \) and \( \Phi_3 \), \( Y = l_1 \cap l_3 \), \( \hat{z} = l_2 \cap \hat{l}_2 \), \( \ddot{z} = l_3 \cap \hat{l}_2 \).

As we have noted previously, \( B \) is the end point of the arc \( FB \) on which the point \( A \) is chosen. Let \( DB \) be another arc of the curve \( \Gamma(\Phi_1) \) that ends at the point \( B \) such that \( \sim FB \cap \sim DB = B \), and \( \delta \) is a sufficiently small positive number. Choose the points \( B_A \in \sim AB \) and \( B_D \in \sim DB \) for which the lengths of the arcs \( BB_A \) and \( BB_D \) do not exceed \( \delta \). It is easy to see that