NONATOMIC PROBLEMS OF LOCATION OF EXTENDED
OBJECTS WITH FINITELY MANY K-EQUIVALENCE CLASSES

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Solution of nonatomic location problems is reduced to minimization of a quadratic form on a closed convex subset in a Hilbert space. Some examples are considered.

Passage to nonatomic problems with a continuum of controlled parameters is a natural approach for qualitative analysis of various problems in which each individual parameter makes an insignificant contribution. Nonatomic models have been developed for transportation problems [1] and in game theory [2]. Location problems [3] are another important class of applications. In this paper, we examine a technique for the analysis of the nonatomic problem of location of extended objects [4].

1. STATEMENT OF THE PROBLEM

Let \( \Omega \) be a continuum of objects to be located, \( \mathcal{A} \) the \( \sigma \)-algebra of its subsets with a given measure \( \mu \), \( K(\omega, \eta) \) an integrable nonnegative bounded symmetric function defined on \( \Omega \times \Omega \). This function is interpreted as the unit cost of interconnection for the objects \( \omega, \eta \in \Omega \) and it is assumed measurable with respect to the standard measure induced on \( \Omega \times \Omega \) by the measure \( \mu \) defined on \( \Omega \). The measure on \( \Omega \times \Omega \) is also denoted by \( \mu \).

The domain \( D \) where the objects from \( \Omega \) are to be located is a closed subset of \( \mathbb{R}^n \) with the usual \( \sigma \)-algebra of measurable sets and the Lebesgue measure \( \nu \). For instance, \( D \) may be a rectifiable curve. The length of interconnections between located objects is measured in the metric \( \rho(\cdot, \cdot) \), which is defined in accordance with the conditions of the problem in the domain \( D \). This metric is assumed continuous in all variables.

A location of \( \Omega \) on \( D \) is an injective mapping \( \psi: \Omega \rightarrow D \) such that the image of any subset \( \Lambda \in \mathcal{A} \) is measurable and \( \mu(\Lambda) = \nu(\psi(\Lambda)) \). The collection of all location mappings is denoted by \( \Phi \). In what follows, we assume that \( \Phi \neq \emptyset \).

Solution of a nonatomic location problem involves finding the value

\[
J = \inf_{\psi \in \Phi} \int_{\Omega} \int_{\Omega} \rho(\psi(\omega), \psi(\eta)) K(\omega, \eta) \mu(\text{d}\omega) \mu(\text{d}\eta),
\]

and the mapping \( \psi^* \) on which the infimum in (1) is achieved. Alternatively, if the infimum in (1) is not achieved, solution of the problem requires finding a minimizing sequence \( \{\psi_n \in \Phi\}_{n=1}^{\infty} \) such that

\[
J = \lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \rho(\psi_n(\omega), \psi_n(\eta)) K(\omega, \eta) \mu(\text{d}\omega) \mu(\text{d}\eta).
\]

Under our assumptions, the problem (1) always has a solution that corresponds to this definition. The examples in Sec. 4 show that the infimum in (1) is not necessarily achieved. In this case, a qualitative view of an optimal solution of the corresponding finite problem can be obtained if we know how to construct a minimizing sequence of locations in the nonatomic problem. The goal of solving the nonatomic problem is thus to find an optimal location if it exists or a minimizing sequence of locations otherwise. In this paper, we consider the conditions of existence of an optimal location in a nonatomic problem.

The problem formulated above can be simplified because some elements of the set $\Omega$ are indistinguishable in a certain sense with respect to the functional (1). The elements $\omega, \eta \in \Omega$ are said to be $K$-equivalent if for any $\gamma \in \Omega$ we have $K(\omega, \gamma) = K(\eta, \gamma)$. This $K$-equivalence relation is obviously reflexive, symmetric, and transitive. Let $\beta = \Omega/K$ be the quotient set of $\Omega$ by $K$-equivalence, $\chi_\gamma(\cdot)$ the characteristic function of the set $\gamma$. The function $K(\omega, \eta)$ is obviously representable in the form

$$K(\omega, \eta) = \sum_{\omega, \eta \in \beta} K_{\omega \eta} \chi_{\omega}(\omega) \chi_{\eta}(\eta),$$

(2)

where summation is over all unordered pairs. The most relevant case in practice is when the set $B$ is finite. This case is analyzed in Sec. 3. Section 2 shows how a general nonatomic problem can be approximated with any prescribed accuracy by a problem with a finite set $B$.

2. APPROXIMATION OF PROBLEM (1) BY A PROBLEM WITH FINITELY MANY $K$-EQUIVALENCE CLASSES

For an arbitrary given $n$, construct the partition $\{k_i \}_{i=1}^N$ of the value set of the function $K(\cdot, \cdot)$ into intervals not longer than $1/n$. Let $\Pi_i^n$ be the preimage of the interval $k_i$ under the mapping $K(\cdot, \cdot)$. Clearly, $\Pi_i^n$ is a symmetric set: $(\omega, \eta) \in \Pi_i^n$ $\Rightarrow$ $(\eta, \omega) \in \Pi_i^n$. The set $\Pi_i^n$ is measurable as the preimage of the measurable set $k_i$ under the measurable mapping $K(\cdot, \cdot)$. Thus, it has a finite symmetric approximation $\Theta_i^n \subseteq \Omega \times \Omega$ such that

$$\Theta_i^n \cap \Theta_j^n = \emptyset, \ i \neq j, \ i, j = 1, 2, ..., N.$$  

(3)

For each $\Pi_i^n$ fix some approximation $\Theta_i^n$, and let $\mathcal{G}_n$ be the family of all subsets from $\mathcal{A}$ used in this approximation of the family $\{\Pi_i^n\}_{i=1}^N$. Consider the set ring $\mathcal{G}(\cdot)$, generated by the family of sets $\mathcal{G}_n \cap \{\Omega\}$. As $B(\cdot)$ take the minimal family of pairwise nonintersecting sets, such that every set from $\mathcal{G}(\cdot)$ is representable as the union of sets from $B(\cdot)$. For the sets $\omega, \eta \in B(\cdot)$ such that $(\omega \times \eta) \in \Theta_i^n$, let

$$K_{\omega \eta}^{B(\cdot)} = \sup_{\omega, \eta \in B(\cdot)} K(\omega, \eta).$$

Setting

$$K^{B(\cdot)}(\omega, \eta) = \sum_{\omega, \eta \in B(\cdot)} K_{\omega \eta}^{B(\cdot)} \chi_{\omega}(\omega) \chi_{\eta}(\eta),$$

we have

$$\mu\left\{(\omega, \eta) \in \Omega \times \Omega \mid |K(\omega, \eta) - K^{B(\cdot)}(\omega, \eta)| > \frac{1}{n}\right\} = \sum_{i=1}^N \mu(\Pi_i^n \Delta \Theta_i^n) < \frac{1}{n}.$$  

The equality in the last relationship holds because by construction $(\omega, \eta) \in \Pi_i^n \cap \Theta_i^n$ implies $|K(\omega, \eta) - K^{B(\cdot)}(\omega, \eta)| \leq 1/n$. The inequality follows from (3).

Note that the quantity $J$ introduced in (1) may be regarded as a functional of $K$. We accordingly use the notation $J(K)$.

Consider the difference $|J(K) - J(K^{B(\cdot)})|$. For definiteness, let $J(K) \geq J(K^{B(\cdot)})$ (the other case is considered similarly). Since

$$J(K^{B(\cdot)}) = \inf_{\psi \in \Phi} \int_{\Omega \times \Omega} \rho(\psi(\omega), \psi(\eta)) K^{B(\cdot)}(\omega, \eta) \mu(d\omega) \mu(d\eta),$$

for any $\varepsilon > 0$ there exists $\psi^* \in \Phi$ such that

$$J(K^{B(\cdot)}) > \int_{\Omega \times \Omega} \rho(\psi^*(\omega), \psi^*(\eta)) K^{B(\cdot)}(\omega, \eta) \mu(d\omega) \mu(d\eta) - \varepsilon.$$  

Thus