Failure of a Brittle Body Close to a Hole
As a Result of Development of a System of Surface Cracks

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An analysis of failures of many constructions showed that a brittle failure, as a rule, begins from a surface, and most frequently from a surface of stress concentrators which are virtually present in all modern constructions. The seats of failure are provided by numerous defects of the surface and, in particular, surface cracks which are the basic cause of reduction of the strength of real bodies.

Close to such stress concentrators as holes there is a zone of increased stresses. This zone is conducive to a further growth of the cracks being formed on the surface; this in the final analysis leads to a total failure of a brittle body.

In [1, 2] it is shown that the dangerous influence of the zone of increased stresses is significant only on comparatively small cracks close to the boundary of the hole. At the same time, if the length of the crack is greater than the characteristic dimension of the hole (the radius for a circular hole), then this influence virtually is absent and the calculation can be carried out without taking the hole into account. This considerably facilitates the solution of the problem.

During recent times there appeared a number of investigations [3] in which the propagation of cracks close to holes was studied. However, the approximate method used in these works did not enable the development of small cracks to be investigated, since the amount of computations steeply increased with reduction in the length of the crack. This leads to the circumstance that at present the case of greatest importance to practice is but little investigated.

In [4] a new approach was proposed for such a class of problems; this approach enables an effective solution to be obtained in closed form.

In the given report, on the basis of this approach, we investigate the problem of limiting equilibrium of a thin plate weakened by a circular hole and a system of n radial small cracks going out to the boundary of the hole, under biaxial tension of the plate by the principal stresses $N_1$ and $N_2$.

1. Problem Formulation. Approximation of the Mapping Function. We consider an infinite elastic plane (plate) weakened by a circular hole of radius $R$ and n symmetrically located straight-lined radial small cracks of length $l$, when "at infinity" the principal stresses $N_1 = p$ and $N_2 = \lambda p$ ($0 < \lambda < 1$) are applied, with the stresses $N_1$ being directed at an angle $\alpha$ to the Ox axis, while the boundaries of the hole and cracks are free from external loads (Fig. 1). We determine the stress state in the elastic plane and the coefficients of stress intensity at the crack tips. Proceeding from there, we investigate the role of the various geometrical and physico-mechanical parameters on the magnitude of the limiting load which gives rise to the beginning of crack development.

The function mapping the exterior of a hole, with n cracks going out to its boundary, in the $z$ plane has the form [1]

$$z = \omega(\zeta) = R^2(\zeta^n + \zeta^{-n} + (1 + \gamma) + (1 - \zeta^{-n})(\zeta^{2n} + 2\zeta^n + 1)^{1/m}. \quad (1)$$

Here
\[ R_n^0 = (1 - \tau)^{-1/\nu}; \]
\[ \tau = 1 - 8(1 + \delta)^n [1 + (1 + \delta)^n]^{-2}; \]
\[ \delta = \frac{1}{R}. \]

The mapping function (1) can be expanded in the series
\[ \omega_N(\zeta) = R_n \left[ \zeta + \sum_{i=1}^{\infty} c_i \zeta^{1-n_s} \right]; \]
\[ R_n = R \left[ 2(1 - \tau)^{-1/\nu} \right], \] (2)
where the \( c_i \) are real coefficients.

The series (2) converges everywhere in \(|\zeta| > 1\), with the exception of the points \( b_n \) of the boundary \( \gamma \) which correspond to the points of intersection of the banks of the crack with the boundary of the hole.

Further, following [1, 2] we approximate the boundary (2) by the expression
\[ \omega_N(\zeta) = R_n \left[ \zeta + \sum_{i=1}^{N} c_i \zeta^{1-n_s} \right] \] (3)
in such a way that the condition
\[ \omega'_N(\zeta) = R_n (1 - \zeta^{-n}) Q_N(\zeta) \] (4)
is satisfied, where \( Q_N(\zeta) \) is a polynomial in negative powers of \( \zeta \); all roots of this polynomial lie within a unit circle in the \( \zeta \) plane. When constructing the function (3) we also stipulate that \( \omega_N(\zeta) \to \omega(\zeta), \omega'_N(\zeta) \to \omega'(\zeta) \) for \( N \to \infty \) everywhere in \(|\zeta| > 1\), with the exception of the points \( b_n \) of the boundary \( \gamma \). We note that the function (3) will map onto \(|\zeta| > 1\) the exterior of a certain new boundary. On this boundary the points of re-entry at the ends of the cracks are retained; only the corners on the intersection at the tip of a crack are rounded, where, by increasing \( N \), we can make the radius of rounding of these corners as small as desired. In this case, if \( \delta = l/R \) is a small quantity, the expression (3) can be represented in the form [4]
\[ \omega_N(\zeta) = R_n \left[ \zeta + \varepsilon \sum_{i=1}^{N} \widetilde{c}_i \zeta^{1-n_s} \right], \] (5)
where \( \varepsilon \) is a small real parameter.

We next transform the function (5) into
\[ \omega_N(\zeta) = R_n \left[ \zeta + \varepsilon \sum_{i=1}^{nN} \widetilde{c}_i \zeta^{1-n} \right]; \]
\[ \widetilde{c}_m = \begin{cases} \frac{0}{m = \sqrt{s n_s}}, \\ \frac{c_s}{m = n_s, s = 1, 2, \ldots, N}. \end{cases} \] (6)

It should be noted that by representing the expression (3) in the form (6) allows us to considerably simplify the solution of the problem thus formulated.

2. Determination of Complex Potentials. The boundary conditions [5] in the case being considered are written as follows:
\[ \varphi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \Psi'(\sigma) + \Psi(\sigma) = 0; \]
\[ \frac{\varphi(\sigma)}{\omega(\sigma)} \Psi'(\sigma) + \Psi(\sigma) = 0. \] (7)

The stress functions corresponding to \( \omega_N(\zeta) \) are denoted by \( \varphi_N(\zeta), \psi_N(\zeta) \).