An estimate is given of the mean value with respect to a Gaussian measure of the distance between distributions of two finite collections of linear functionals over a Euclidean space with probability measure for certain metrics in the space of finite-dimensional distributions.

This paper is devoted to the study of joint distributions of all collections of linear functionals, defined on a Euclidean space with measure.

Let \((X_N, I_N), (X^*_N, I^*_N)\) be \(N\)-dimensional Euclidean spaces in duality. By \(\mathcal{P}^N(C_0)\) we denote the class of Borel probability measures \(\mathcal{P}\) on \(X^*_N\), satisfying the following conditions:

\[
\int X^*_N \mathcal{P} \leq c_0 \|\mathcal{P}\|_X \quad \text{for all } \mathcal{P} \in \mathcal{P}^N(C_0).
\]

The joint distribution of the functionals \(f_1, \ldots, f_K\) from \(X^*_N\) with respect to the measure \(\mathcal{P}\) on \(X^*_N\) will be denoted by \(\mathcal{P}_{f_1, \ldots, f_K}\).

We shall be interested in estimates of the mean value of the distance between \(\mathcal{P}_{f_1, \ldots, f_K}\) and \(\mathcal{P}_{g_1, \ldots, g_K}\) with respect to a given metric. The analogous problem for distributions of linear functionals was considered in [2-4] and was used in [4] to get a new proof of the typical distribution theorem of [1].

We introduce some definitions and notation.

1. Let \(\mu\) be a probability measure on \(\mathbb{R}^r\), \(t=(t_1, \ldots, t_r) \in \mathbb{R}^r\). Let \(I_t=\{i_1, \ldots, i_k\}\) be a collection of indices such that \(t_{i_l}>0\) for \(l \in I_t\) and \(t_{i_l}<0\) for \(l \notin I_t\). We set

\[
\mathfrak{F}_{\mu}^r(t) = \mu\left(\{s=(s_1, \ldots, s_r) \in \mathbb{R}^r \mid \left(\sum_{l \in I_t} s^2_{i_l} \right)^{1/2} > 0\right) \quad \text{for } t \in I_t.
\]

We note that if \(k=1\) the distribution function of the measure \(\mu\) coincides with \(\mathfrak{F}_{\mu}^r\) for \(t<0\) and with \(1-\mathfrak{F}_{\mu}^r\) for \(t>0\).

2. Let \(\alpha\) be a real number; \(\mu, \nu\) be probability measured on \(\mathbb{R}^r\) such that all the coordinate functionals are square summable with respect to each of these measures. Let \(\mathfrak{F}_{\mu}, \mathfrak{F}_{\nu}\) be the functions defined by (1). We define a metric \(\rho_{\alpha}\) by the following equation:

\[
\rho_{\alpha}(\mu, \nu) = \left(\int_{\mathbb{R}^r} |\mathfrak{F}_{\mu}(b) - \mathfrak{F}_{\nu}(b)|^\alpha \, db \right)^{1/\alpha}.
\]

The metric \(\rho_{\alpha}\) is a generalization of the metric \(\mathfrak{A}_0\), considered in [3] for one-dimensional distributions and defined by the equation:

\[
\mathfrak{A}_0(\mu, \nu) = \left(\int_{-\infty}^{\infty} |\mathfrak{F}_{\mu}(b) - \mathfrak{F}_{\nu}(b)| \, db \right)^{1/2}.
\]

3. It is easy to see that the metric \(\rho_{\alpha}\) depends on the coordinate system. It is easy to get rid of this deficiency by introducing the metric \(\overline{\rho}_{\alpha}\):
where the integration is with respect to the normalized Haar measure on the group $O(k)$ of orthogonal transformations of the space $\mathbb{R}^k$.

4. By $\mathcal{N}$ we shall denote the standard Gaussian measure in $(X_1^k, \mathcal{B}_1)$; by $\mathcal{N}_t^k$ the standard Gaussian measure in $(X_t^k)^{\times k}$ (k times).

Remark 1. Let $\mu^t, \nu$ be probability measures in $\mathbb{R}$. If $\int_{\mathbb{R}^k} \|t\|^2 d\mu(t) < +\infty$ and $\int_{\mathbb{R}^k} \|t\|^2 d\nu(t) < +\infty$, then $\mu^t \ast \nu < +\infty$ for $a \in (-t, \frac{1}{2} - t)$.(For the proof it suffices to show that $\int_{\mathbb{R}^k} (\|t\|^2)^{\frac{1}{2}} d\mu(t) < +\infty$.)

Remark 2. The condition $k(a+1) \leq 4$ is essential for the finiteness of the metric $\rho_a$. For example if $k=2$, $a > 4k-1$, and the measure $\nu$ in $X^k$ is defined by the density $e^{-\|x\|^2}$ for $\|x\| < 1$, $e^{-\|x\|^2}$ for $\|x\| > 1$.

Remark 3. The estimates of Theorem 1 remain valid upon replacing $\mathcal{E}_a^N(P, \nu)$ by $e_\sigma^N(P_k, \nu) = \int_{\mathbb{R}}^\delta \mathcal{K}_a^N \left( P_k, \nu, \|x\| < \delta, \|x\| < 1 \right)$ for all $P \in \mathcal{E}_a^N$, $\|x\| < 1$, where $e_\sigma^N$ is the Gaussian measure on $(X_1^k, \mathcal{B}_1)$ gotten from the standard compression by $\mathcal{N}$ times. Here the dimension $N_\sigma(a, k)$ can be estimated as follows:

$$N_\sigma(a, k) = 0(c_0^2 \|x\| + \frac{1}{a^2})^2, \quad \text{for } 0 < k(a+1) < 2,$$

$$N_\sigma(a, k) = 0(c_0^2 \|x\| + \frac{1}{a^2})^2, \quad \text{for } 2 < k(a+1) < 4.$$