OPTIMAL INVENTORY MANAGEMENT UNDER INCOMPLETE DEMAND INFORMATION

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A single-period, single-commodity inventory management model is considered. A minimax-cost ordering rule is derived for the case when only the mean and the variance of the demand distribution function are known.

1. INTRODUCTION

The optimal single-period volume of deliveries \( Q \) minimizing the mean losses in a system that sells goods from stock was derived in [1] assuming that the demand distribution function was unknown and only its first two moments \( s_1 \) and \( s_2 \) were known. However, the solution obtained in [1] is incomplete. Thus, Theorem 2 [1, p. 255] is true only for \( Q > s_2/2s_1 \). Moreover, it is assumed in [1] that the maximum demand and the maximum inventory level are unbounded (or unknown), not a very realistic assumption.

In this paper, we derive a complete solution of the problem by a method that differs from that in [1]. We also examine the case of bounded demand. Our paper only considers a single period of deliveries and sales, which is treated in isolation. Such formulations have been used in previous applications, e.g., in [4].

2. STATEMENT OF THE PROBLEM AND REVIEW

We assume that goods can be delivered to inventory only at the beginning of the period and that \( Q \) is the unknown volume of deliveries. The demand for these goods is a random variable \( X \), \( 0 \leq X \leq A \) (\( A \) is the maximum allowed demand, \( 0 < A < \infty \)) with an unknown distribution function \( F(x) = P\{X < x\} \). We only know that the distribution function \( F(x) \) is from the class \( K \):

\[
K = \left\{ F : F(0^-) = 0, \quad F(A) = 1, \quad \int_0^A x\,dF(x) = s_1, \quad \int_0^A x^2\,dF(x) = s_2, \quad 0 < s_1 < A, \quad s_1^2 < s_2 < s_1A \right\}.
\]

The maximum inventory does not exceed the maximum demand: $0 \leq Q \leq A$. The following pricing system is given:

$q$ is the loss associated with each unit of unsold inventory; $p$ is the penalty for each unit of unsatisfied demand, $p, q > 0$. Then the mean loss $ML$ in the system during the entire period is

$$ML = q \int_0^Q (Q - x) \, dF(x) + p \int_Q^A (x - Q) \, dF(x) =$$

$$= (p + q) \int_0^Q (Q - x) \, dF(x) + p (s_1 - Q).$$

(1)

The problem is to find

$$Q^* = \arg \min_{Q \in K} \sup_{F \in K} ML$$

as a function of the parameters $q, p, A, s_1, s_2$.

The solution of this problem is stated at the end of the paper in the form of Theorem 2 and a corollary. Section 3 gives the "worst" demand distributions which result in highest mean losses. Section 4 presents the necessary background for proving the bounds of Sec. 3 and Sec. 5 gives the proof. Finally, Sec. 6 minimizes the maximum mean loss and derives optimal rules for the determination of the inventory level $Q$ as a function of the parameters.

3. EXACT UPPER BOUNDS ON MEAN LOSSES

Let $A < \infty$. Then the following upper bounds hold.

**Bound 1.** For $Q$ in the interval $[0; s_2/2s_1)$,

$$\sup_{F \in K} \int_0^Q (Q - x) \, dF(x) = \frac{Q (q \sigma^2) / s_3}{s_2}$$

and this bound is attained on a two-step distribution function $F_1 \in K$ with points of increase $x_1 = 0$ and $x_2 = s_2/s_1$; $\sigma^2 = s_2 - s_1^2$.

**Bound 2.** For $Q$ in the interval $(s_2/2s_1; A_1)$, where $A_1 = (A^2 - s_2) /[2(A - s_1)]$,

$$\sup_{F \in K} \int_0^Q (Q - x) \, dF(x) = \frac{1}{2} \left( Q - s_1 + \sqrt{Q^2 - 2Qs_1 + s_2^2} \right)$$

and this bound is attained on a two-step distribution function $F_2 \in K$ with points of increase $x_1^0 = Q - [(Q - s_1)^2 + \sigma^2]^{1/2}$ and $x_2^0 = Q + [(Q - s_1)^2 + \sigma^2]^{1/2}$.

**Bound 3.** For $Q$ in the interval $(A_1, A)$,

$$\sup_{F \in K} \int_0^Q (Q - x) \, dF(x) = (A - s_1) \frac{QA - s_1 (Q + A) + s_3}{A^2 - 2As_1 + s_2}$$

and this bound is attained on a two-step distribution function $F_3 \in K$ with points of increase

$$y_1 = \frac{s_1 A - s_3}{A - s_1}, \quad y_2 = A.$$

If $A = \infty$, then bounds 1 and 2 hold with $\infty$ substituted for $A$. These bounds were derived by the author in 1978 [2]. From bounds 1-3 and (1), we obtain the maximum mean losses as a function of $Q$.

1. If $0 \leq Q < s_2/2s_1$, then

$$\sup_{F \in K} ML = Q (q \sigma^2 - ps_3) / s_3 + ps_4 = \Phi_1(Q).$$

2. If $s_2/2s_1 < Q < (A^2 - s_2) /[2(A - s_1)]$, then

$$\sup_{F \in K} ML = \frac{p + q}{2} \left( Q - s_1 + \sqrt{(Q - s_1)^2 + \sigma^2} \right) + p (s_1 - Q) = \Phi_2(Q).$$