ASYMPTOTIC BEHAVIOR OF THE LOGARITHMIC DERIVATIVE OF AN ENTIRE FUNCTION OF COMPLETELY REGULAR GROWTH

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In [1] the authors found asymptotic expressions for the logarithmic derivative of an entire function of completely regular growth in the sense of Levin and Pfuger [2]. These asymptotic expressions hold outside of a certain exceptional set which (just as in the general case of exceptional sets in the theory of entire functions of completely regular growth) was not effectively constructed. Levin [2] distinguished a subclass of entire functions of completely regular growth for which it is possible to find the asymptotic behavior outside of a certain effectively defined system of circles. Here we show that outside of this same system of circles we have an asymptotic equation also for the logarithmic derivative of an entire function of the same subclass.

Let \( f(z) \) be an entire function of order \( \rho > 0 \) and of completely regular growth relative to the proximate order \( \rho(r) \sim \rho \) as \( r \to \infty \). Let \( \{a_k\} \) be the sequence of zeros of the function \( f(z) \). Following Levin, we say that \( \{a_k\} \) is an R-set if there exists a number \( d > 0 \) such that the disks

\[
U_k = \{z : |z - a_k| < d |a_k|^{1-\rho(d/2)}\}, \quad k = 1, 2, \ldots ,
\]

do not intersect, and the set \( U = \bigcup_{k=1}^{\infty} U_k \) will be called a set of \( C_R \)-disks.

**Theorem 1.** Let \( f(z) \) be an entire function of nonintegral order \( \rho \) and of completely regular growth relative to the proximate order \( \rho(r) \sim \rho \) as \( r \to \infty \) and whose zeros form an R-set. Let \( \Delta(\varphi) \) be the angular density function of the sequence of zeros of \( f(z) \), \( E \) the set of points of discontinuity of the function \( \Delta(\varphi) \), \( M \) an arbitrary closed set, \( M \subset [0, 2\pi] \setminus E \), \( U \) the set of \( CR \)-disks. Then for \( z = re^{i\varphi} \in \mathbb{C} \setminus U \) we have, uniformly with respect to \( \varphi, \varphi \in M \), that

\[
\frac{f'(z)}{f(z)} = \mathcal{H}(\varphi) r^{\rho(\pi - 1)} + o(r^{\rho(\pi - 1)}), \quad r \to \infty,
\]

where

\[
\mathcal{H}(\varphi) = -\frac{\pi \rho}{\sin \rho \pi} \exp \left\{ i \left( -\varphi + (\rho - 1) (|\varphi - \varphi| - \pi) \text{sgn}(\varphi - \varphi) \right) \right\} d\Delta(\varphi).
\]

This theorem differs from Theorem 1 in [1] by the fact that in the latter it is not required that the zeros of \( f(z) \) form an R-set, and (1) holds not outside \( U \), but outside some system of circles of zero \( \mu \)-density, \( 1 < \mu \leq 2 \). The case of integral order will be considered at the end of the paper, and we now proceed to the proof of Theorem 1.

Without loss of generality, we can assume that \( f(z) \) is a canonical Weierstrass product of genus \( p = [\rho] \). By \( n(r) \) we denote the counting function of the sequence of zeros of \( f(z) \), and by \( \Delta \) the density of the sequence \( \{a_k\} \) relative to the proximate order \( \rho(r) \).

**Lemma 1.** Let \( \{a_k\}_{k=1}^{N} \) be a set of \( N \) distinct complex numbers, and the number \( \tau > 0 \) such that the disks

\[
U_k^\tau = \{z : |z - a_k| < \tau\}
\]

are pairwise disjoint. Then for \( z \in U^\tau = \bigcup_{k=1}^{N} U_k^\tau \) we have the inequality

\[
\sum_{k=1}^{N} \frac{1}{|z - a_k|} < \frac{9}{\pi} \sqrt{1 + \frac{N}{3}}.
\]
Proof. We fix \( z \in U_T \). Suppose that the ring \( \{ \xi : |\xi - z| < (j + 1)T \} \) contains \( s_j \) points of \( \{ a_k \}_{k=1}^N \), \( j = 1, 2, \ldots, q \), and that in \( \{ \xi : |\xi - z| \geq (q + 1)T \} \) there are no points of \( \{ a_k \}_{k=1}^N \). Since \( z \in U_T \), in the disk \( \{ \xi : |\xi - z| < T \} \) there are also no points of \( \{ a_k \}_{k=1}^N \). Obviously, \( \sum_{j=1}^q s_j = N \). Since the disks \( U_k \) are pairwise disjoint, for \( 1 \leq j \leq q \) we have

\[
s_j \leq S_j := \frac{\pi r^2 (j + 2)^2 - \pi r^2 (j - 1)^2}{\pi r^2} = 3(j + 1).
\]

Obviously,

\[
\sum_{j=1}^N \frac{1}{|z - a_k|} = \sum_{j=1}^q \left( \sum_{k:|a_k-z|<(j+1)T} \sum_{k:|a_k-z|>=(q+1)T} \frac{1}{|z - a_k|} \right) \leq \frac{1}{r} \sum_{j=1}^q s_j.
\]

(3)

We assume that \( s_j = S_j \) for \( 1 \leq j \leq q \), \( l \geq 0 \), and \( s_{l+1} < S_{l+1} \) and \( s_m > 0 \), \( s_j = 0 \) for \( m+1 \leq j \leq q \), \( m \leq q \). Then if \( m > l + 1 \),

\[
\sum_{j=1}^m s_j = \sum_{j=1}^l S_j + \sum_{j=l+1}^m \frac{s_j}{j} < \sum_{j=1}^l S_j + \frac{s_{l+1} + 1}{l+1} + \sum_{j=l+2}^m \frac{s_j}{j} + \frac{s_m - 1}{m} = \sum_{j=1}^m \frac{s_j}{j},
\]

(4)

where \( \sum_{j=1}^m s_j = N \), \( s_j \leq S_j \), \( 1 \leq j \leq m \). If \( s_{l+1} + 1 < S_{l+1} \), then the right-hand side of (4) can be estimated as earlier; if \( s_{l+1} + 1 = S_{l+1} \), then we proceed in exactly the same way, but we consider the number \( l \) to be larger than 1.

In both cases, if \( s_m = 1 \), then the number \( m \) is replaced by a smaller one. Repeating these arguments a finite number of times, we see that there exists a \( v \geq 1 \) such that

\[
\sum_{j=1}^q s_j = \sum_{j=1}^{v-1} S_j + S_v + \frac{s_v}{v},
\]

(5)

where \( 0 \leq v < S_v \) and \( \sum_{j=1}^{v-1} S_j + S_v = N \). Further,

\[
N > \sum_{j=1}^l S_j = \sum_{j=1}^{v-1} 3(2j + 1) = 3(v^2 - 1), \quad v < \sqrt{\frac{1 + N}{3}}.
\]

(6)

On the other hand,

\[
\frac{S_j}{j} = 3 \left( 2 + \frac{1}{j} \right) < 9, \quad j > 1.
\]

(7)

Therefore, taking (5)-(7) into account, we get from (3) that

\[
\sum_{k=1}^N \frac{1}{|z - a_k|} < \frac{1}{r} \sum_{j=1}^q \frac{S_j}{j} < \frac{9}{r} v < \frac{9}{r} \sqrt{1 + \frac{N}{3}}.
\]

LEMMA 2. Suppose that the conditions of Theorem 1 hold. We use the notation \( (0 < \beta \leq 1/2) \)

\[
A(\phi, \beta, r) = \int_{1/(1+\beta)}^{1+\beta} \frac{e^{(r-\phi)\beta - 1}}{t - e^{\phi}} dt,
\]

(8)

\[
B(\phi, \beta, r) = \int e^{\omega(\phi, \beta) - \phi} e^{\beta r} \, d\Delta(\phi).
\]

(9)

where \( \omega(\phi, \psi) = \phi - \psi \) if \( \psi > \phi \geq 2\pi \), and \( \omega(\phi, \psi) = 2\pi - (\psi - \phi) \) if \( 0 \leq \phi > \psi \). Then for any \( \varepsilon > 0 \) there exist \( \beta_0 > 0 \) and \( r_0 > 0 \) such that for all \( \beta, 0 < \beta < \beta_0 \), all \( r, r > r_0 \), and all \( \phi \in \Delta \) we have

\[
|B(\phi, \beta, r)| < \varepsilon.
\]

(10)